## Reasoning and Proof Standard for Grades 6–8

**Instructional programs from prekindergarten through grade 12 should enable all students to—**

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Reasoning is an integral part of doing mathematics. Students should enter the middle grades with the view that mathematics involves examining patterns and noting regularities, making conjectures about possible generalizations, and evaluating the conjectures. In grades 6–8 students should sharpen and extend their reasoning skills by deepening their evaluations of their assertions and conjectures and using inductive and deductive reasoning to formulate mathematical arguments. They should expand the audience for their mathematical arguments beyond their teacher and their classmates. They need to develop compelling arguments with enough evidence to convince someone who is not part of their own learning community.

### What should reasoning and proof look like in grades 6 through 8?

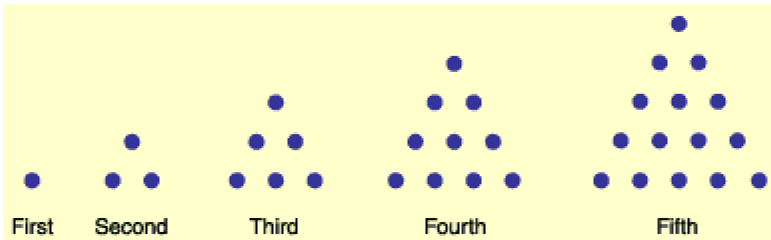
In the middle grades, students should have frequent and diverse experiences with mathematics reasoning as they—

- examine patterns and structures to detect regularities;
- formulate generalizations and conjectures about observed regularities;
- evaluate conjectures;
- construct and evaluate mathematical arguments.

Students should discuss their reasoning on a regular basis with the teacher and with one another, explaining the basis for their conjectures and the rationale for their mathematical assertions. Through these experiences, students should become more proficient in using inductive and deductive reasoning appropriately.

Students can use inductive reasoning to search for mathematical relationships through the study of patterns. Consider an example from a classroom in which rising seventh-grade students were studying figurate numbers (drawn from classroom observation and partially described in Malloy [1997]).

The teacher began by explaining triangular numbers and then asked the students to generate representations for the first five triangular numbers. The students visualized the structure of the numbers to » draw successive dot triangles, each time adding at the bottom a row containing one more dot than the bottom row in the previous triangle (see fig. 6.31). Next the teacher asked the students to predict (without drawing) how many dots would be needed for the next triangular number. Reflecting on what they had done to generate the sequence thus far, they quickly concluded that the sixth triangular number would have six more dots than the fifth triangular number. These students were engaged in recursive reasoning about the structure of this sequence of numbers, using the just-formed number to generate the next number. This approach was repeated for several more "next" numbers in the sequence, and it worked well.




The teacher then asked the students to find the 100th term in the sequence. Most students knew that the value of the 100th term is 100 more than the value of the 99th term, but because they did not already know the value of the 99th term, they were not able to find the answer quickly. The teacher suggested that they make a chart to record their observations about triangular numbers and to look for a pattern or a relationship to help them find the 100th triangular number. The students began with a display that reflected what they had already observed (see fig. 6.32). They examined the display for additional patterns. Tamika commented that she thought there was a pattern relating the differences and the numbers. She explained that if the consecutive differences are multiplied, the product is twice the number that is "between" them in the display; for example, the product of 4 and 5 is twice as large as 10.

| Term | First | Second | Third | Fourth | Fifth | Sixth |
|---|---|---|---|---|---|---|
| Number | 1 | 3 | 6 | 10 | 15 | 21 |
| Difference | | 2 | 3 | 4 | 5 | 6 |

The teacher asked the students to check to see if Tamika's observation was true for other numbers in the display. After they verified the observation, the teacher asked them to use this method to find the next triangular number. Some students were unable to see how it could be done, but Curtis used Tamika's observation as follows: "Using Tamika's method, the seventh number is (7)(8)/2, which is 28." Several students checked this answer by using the recursive method of adding 7 to the sixth triangular number to find the seventh triangular number (21 + 7 = 28). The teacher then asked the students to check Tamika's method for the next few triangular numbers to verify that it worked in those instances. She next asked if

Tamika's method could be used to find the 100th triangular number. Darnell said, "If Tamika is right, the hundredth triangular number should be (100)(101)/2."

In general, the students agreed that the method of multiplying and dividing by 2 was useful because it seemed to work and because it did not require knowing the *n*th term in order to find the (*n* + 1)th term. However, some students were not convinced that the method was correct. It lacked the intuitive appeal of the recursive method they used first, and it did not appear to have a mathematical basis. The teacher decided that it was worth additional class time to develop a mathematical argument to support Tamika's method. She began by asking students to notice that each triangular number is the sum of consecutive whole numbers, which they readily saw from the dot triangles. Then the teacher demonstrated Gauss's method for finding the sum of consecutive whole numbers, applying it to the first seven whole numbers. She asked the students to add the numbers from 1 to 7 to those in the reversed sequence, 7 to 1, as shown in figure 6.33, to see that the seventh triangular number—1 + 2 + 3 + 4 + 5 + 6 + 7—could also be expressed as (7)(8)/2. After the students completed this exercise, the teacher asked them to express the general relationship in words. They struggled, but they came up with this general rule: If you want to find a particular triangular number, you multiply your number by the next number and divide by 2. The students wrote the rule this way:
(number)(number + 1)/2.

| | |
|---|---|
| $1+2+3+4+5+6+7$<br>$7+6+5+4+3+2+1$ | Students can see that the sums of the pairs of addends can be represented as 7 × 8, or 56. |
| $8+8+8+8+8+8+8$ | Because each number is listed twice, they divide 56 by 2, resulting in (7)(8)/2 = 56/2 = 28. |

The example illustrates what reasoning and proof can look like in the middle grades. Although mathematical argument at this level lacks the formalism and rigor often associated with mathematical proof, it shares many of its important features, including formulating a plausible conjecture, testing the conjecture, and displaying the associated reasoning for evaluation by others. The teacher and students used inductive reasoning to reach a generalization. They noted regularities in a pattern (growth of triangular numbers), formulated a conjecture about the regularities (Tamika's rule), and developed and discussed a convincing argument about the truth of the conjecture.

Middle-grades students can develop arguments to support their conclusions in varied topics, such as number theory, properties of geometric shapes, and probability. For example, students who encounter the rules of divisibility by 2 and by 3 in number theory know that even numbers are divisible by 2 and numbers whose digits add to a number divisible by 3 are divisible by 3. A teacher might ask students » to formulate a rule for divisibility by 6 and develop arguments to support their rule.

Some students might begin by listing some multiples of 6: 12, 18, 24, and 30.

They could examine the numbers and try to detect patterns resembling those in other rules they have learned. Students might observe that all the numbers are even, which allows them to infer divisibility by 2. They could also look at the sums of the digits of the multiples and notice that the sums of the digits are all divisible by 3, just as in the test for divisibility by 3. Noting that $2 \cdot 3 = 6$, they might conclude that if the number is divisible both by 2 and by 3, then it must be divisible by 6, which might lead them to form the following conjecture for determining whether a number is divisible by 6: Check to see if the number is even and if the sum of its digits is divisible by 3.

The teacher should also challenge students to consider possible limitations of their reasoning. For example, she could ask them to use 12 as an example to consider whether it is always true that the product of two factors of a number is itself a factor of that number. The students should note that although 6 and 4 are both factors of 12, $6 \cdot 4$ is not. In this way, the teacher can help students become appropriately cautious in making inferences about divisibility on the basis of factors. Such an exploration should lead to the correct generalization that combining criteria for divisibility, which worked with divisibility by 6, works only when the two factors are relatively prime.