Lecture 5: Spatial Algorithms

GEOG 419: Advanced GIS

Spatial Analysis Algorithms

• Basis of much of GIS analysis today
• Manipulation of map coordinates
• Based on Euclidean coordinate geometry

• http://astronomy.swin.edu.au/~pbourke/geometry/

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A matter of analytical cartography

• Theoretical and mathematical background behind cartography
• Seeks to find how geographic properties of space can be used in analysis, modeling, and prediction
• Analytical Cartography consists of the basic mathematical algorithms and principles of cartography that survive independently of a particular technology

AC Algorithms

• **Algorithm** – special method for solving a problem stated as a formula or a set of sequential instructions
• **Church’s Theorem** – if a problem can be stated as a series of sequential instructions, then it can be automated
• Cartographic transformational algorithms are the nuts and bolts from which GIS are constructed
Matrices

- If you are going to understand AC algorithms, you should understand something about matrix algebra.
- **Matrix**: a set of numbers of symbols arranged in a square or rectangular array of “m” rows and “n” columns. The arrangement is such that certain defined mathematical operations can be performed in a systematic and efficient way.

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Linear equations and matrices

Given a set of linear equations

\[
\begin{align*}
 a_1 x_1 + a_2 x_2 + a_3 x_3 &= c_1 \\
 a_2 x_1 + a_2 x_2 + a_2 x_3 &= c_2 \\
 a_3 x_1 + a_3 x_2 + a_3 x_3 &= c_3
\end{align*}
\]

\[
\begin{align*}
 x + y - z &= -8 \\
 2x - y + z &= -4 \\
 -x + 2y + 2z &= 21
\end{align*}
\]

We can define them in matrix form

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3
\end{bmatrix}
= 
\begin{bmatrix}
 c_1 \\
 c_2 \\
 c_3
\end{bmatrix}
\]

And further define them in matrix notation

\[
AX = C
\]
• So, if we have a system of linear equations

\[ AX = C \]

• We can algebraically reformat the equation by pre-multiplying both sides by the identity matrix

\[
\begin{align*}
AX &= C \\
A^{-1}X &= A^{-1}C \\
IX &= A^{-1}C \\
X &= A^{-1}C
\end{align*}
\]

Solving simultaneous equations

• We can solve the following equation simultaneously through based algebraic manipulation

\[
\begin{align*}
\text{for } y \text{, we reduce to } \\
x + 5y + 8 &= -x - 2y + 1 \\
\text{now add the two equations } \\
x + 5y + 8 \\
-x - 2y + 1 \\
\hline
3y &= -9 \\
y &= -3
\end{align*}
\]
Solving simultaneous equations with matrices

• Or, we can use matrices

\[
\begin{align*}
x + 5y &= -8 \\
-x - 2y &= -1
\end{align*}
\]

\[
\begin{bmatrix}
1 & 5 \\
-1 & -2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
-8 \\
-1
\end{bmatrix}
\]

More complicated formulas

\[
\begin{align*}
x + 5y &= -8 \\
x + y - z &= -8 \\
2x - y + z &= -4 \\
x + 2y + 2z &= 21
\end{align*}
\]
Transformational View of Cartography

• Types of data/types of maps
• Map scale
• Dimensional
• Symbolization
• Generalization
• Data model

Transformations

• Goal is to express a transformation as an explicit math operation so that \(0 \to 1\) in a fully described way and an inverse transformation \(1 \to 2 = 0\)
• Invertible transformations
  – Special subset, allows for prediction of error, spatial modeling
• Point-to-point transformations – very central, more than any other
• Can have multi-step transformations
Dimensional Transformations

- Coordinate transformations
  - Projections
- Geometric transformations
  - Measurements from coordinates
- Affine transformations
  - Rotation, translation, and scaling
- Statistical space transformations

Planar map transformations

- Distance between two points
  - Figured out 2,400 years ago by Pythagoras
  \[ d_{2\to1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
  - Is this invertible? Only if we have three points
- Length of a line
  \[ length = \sum_{i=1}^{n-1} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \]
Planar map transformations

II

- Weighted average point
- Cities \((x,y,P)\)  \(P = \text{population}\)

\[
\bar{x} = \frac{\sum_{i=1}^{npts} P_i x_i}{\sum_{i=1}^{npts} P_i}
\]

\[
\bar{y} = \frac{\sum_{i=1}^{npts} P_i y_i}{\sum_{i=1}^{npts} P_i}
\]

Intersection point of two lines

\((x_1,y_1)\)
\((x_2,y_2)\)
\((x_3,y_3)\)
\((x_4,y_4)\)

\((p,q)\)
Intersection point of two lines

• If \((x_1, y_1)\) and \((x_2, y_2)\) lie on the same line then:
  \[
  y_1 = a_1 + b_1 x_1 \\
  y_2 = a_1 + b_1 x_2
  \]
  \[
  \text{remember } y = mx + b \text{ (the equation of a line)}
  \]

• If \((x_3, y_3)\) and \((x_4, y_4)\) lie on the same line then:
  \[
  y_3 = a_2 + b_2 x_3 \\
  y_4 = a_2 + b_2 x_4
  \]

If an intersection exists, it must lie on both lines:

\[
\begin{align*}
  y &= a_1 + b_1 x \\
  y &= a_2 + b_2 x
\end{align*}
\]

• Solve for simultaneous equations:

\[
 y - y = a_1 - a_2 + x(b_1 - b_2)
\]

• Rearrange to get:

\[
 a_1 - a_2 = x(b_1 - b_2)
\]
Intersection point of two lines

• Solving for $x$:

\[
x = \frac{a_1 - a_2}{b_1 - b_2}
\]

• And by substituting, solve for $y$:

\[
y = a_1 + b_1 \frac{(a_1 - a_2)}{(b_1 - b_2)}
\]

An example of line intersection
Intersection of two lines

- If the denominator above is 0, the lines are parallel
- If the denominator and the numerator above is 0, the lines are coincident
- In the computer, we have a real problem with 0
  - Actual zero almost never happens
  - Means we must check within limits of coordinate precision
- Solution is not elegant – test every combination of line segments
  - Bounding Box heuristic

\[
x_1 = 50
y_1 = 70
x_2 = 20
y_2 = 10
\]

\[
y_1 = a_1 + b_1 x_1 \quad y_3 = a_2 + b_2 x_3
y_2 = a_1 + b_2 x_2
y_4 = a_2 + b_2 x_4
70 = a_1 + 50b
10 = a_1 + 20b
60 = 30b
b = 2
70 = a_1 + 40
a_1 = -30
10 = a_2 + -120
a_2 = 130
\]

\[
x = \frac{160}{6}
x = 26.667
\]

\[
= -30 + \frac{30 - 130}{2 - 4}
= -23.33
\]
Distance between a point and a line

- Equation for the line
  \[ P = P_1 + u(P_2 - P_1) \]

- The shortest distance from \( P_3 \) to \( P \) is a perpendicular line. This means the dot product of the line and the perpendicular is 0.
  \[ (P_3 - P) \cdot (P_2 - P_1) = 0 \]

- Substitute for the equation of the line
  \[ [P_3 - P_1 - u(P_2 - P_1)] \cdot (P_2 - P_1) = 0 \]
Distance between point and line

- Solve for $u$
  \[ u = \frac{(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)}{\left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)^2} \]

- Find the intersection point
  \[ x = x_1 + u(x_2 - x_1) \]
  \[ y = y_1 + u(y_2 - y_1) \]

- Distance is length of line between P3 and intersection point

Clockwise vs. counter clockwise

What if it's on the line?
Area of a polygon

\[ A = \frac{1}{2} \sum_{i=1}^{n+1} (x_i y_{i+1}) - (x_{i+1} y_i) \]

- Project lines from each vertex to some outside perpendicular. Each area under the line is a triangle and a rectangle. Sum the areas as you move around the polygon. Outside areas get subtracted out.

\[ \sum_{i=1}^{n+1} (x_i y_{i+1}) - (x_{i+1} y_i) \]

\[ \frac{1}{2} \sum_{i=0}^{N} (x_i y_{i+1} - y_{i+1} x_i) \]

\[
\begin{array}{c|c|c}
\text{STATION} & \text{COORDINATES} & \text{AREA} \\
\hline
A & 591.64 & 0 \\
B & 847.60 & 125.66 \\
C & 694.07 & 716.31 \\
D & 0 & 523.62 \\
A & 591.64 & 0 \\
\hline
\end{array}
\]

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What we have covered this semester

• Brief and fast review of GIS
  – Structure of GIS (HW/SW, file based, object based, object-relational) GIS data models (raster, vector, TIN)

• Object modeling and geodatabases
  – Geo-relational model, object oriented concepts, object behavior, benefits of geodatabases
  – Making features smart, qualities of features in a geodatabase, subtype, relationship classes

• Cartographic modeling
  – General principles in cartographic modeling, transfer of principles to Model Builder, spatial SQL

• Database design
  – Data models, entity-relationship diagrams, spatial entities in E-R diagrams, physical and logical design, design steps (model user view, define entities and relationships, transition to physical model)
  – ESRI Geodatabase structure (geodatabase, feature dataset, feature and object classes, relationship classes, rules)

What we have covered this semester

• Vector GIS operations
  – Vector functions (scale change, buffer, clip, overlay, Theissen polygons, Delaunay triangulation)
  – Map transformations (conformal, affine, polynomials, rubbersheeting), raster transformation (WTC example)
  – Spatial algorithms (concept of an algorithm, mathematics of analytical cartography: distance, intersection, containment)
How you got your hands dirty

- Review of classic GIS functions for site selection
- Getting real world data into GIS (even when the data isn’t very good)
  - rubbersheeting and raster transformation
- Integration of classic GIS functions in a modern computing environment (model builder)
- Creating a geodatabase (controlled environment)
- Creating a geodatabase (non controlled environment)

What’s on the written exam

- 10 definitions
- 3 short answers (lists, benefits/limitations, etc.)
- 1 computation (i.e. area, centroid, distance, intersection)
- 2 essays (conceptual ideas)
What’s on the practicum

• Hands-on work
  – Create a geodatabase
    • Digitize boundaries
  – Create a small GIS analysis