Lecture 5:
Spatial Algorithms

GEOG 419: Advanced GIS

Spatial Analysis Algorithms

- Basis of much of GIS analysis today
- Manipulation of map coordinates
- Based on Euclidean coordinate geometry

- http://astronomy.swin.edu.au/~pbourke/geometry/

A matter of analytical cartography

- Theoretical and mathematical background behind cartography
- Seeks to find how geographic properties of space can be used in analysis, modeling, and prediction
- Analytical Cartography consists of the basic mathematical algorithms and principles of cartography that survive independently of a particular technology

AC Algorithms

- **Algorithm** – special method for solving a problem stated as a formula or a set of sequential instructions
- **Church's Theorem** – if a problem can be stated as a series of sequential instructions, then it can be automated
- Cartographic transformational algorithms are the nuts and bolts from which GIS are constructed

Matrices

- If you are going to understand AC algorithms, you should understand something about matrix algebra
- **Matrix**: a set of numbers of symbols arranged in a square or rectangular array of "m" rows and "n" columns. The arrangement is such that certain defined mathematical operations can be performed in a systematic and efficient way

Linear equations and matrices

Given a set of linear equations

$$\begin{aligned} a_1x_1 + a_1x_2 + a_1x_3 &= c_1 \\ a_2x_1 + a_2x_2 + a_2x_3 &= c_2 \\ a_3x_1 + a_3x_2 + a_3x_3 &= c_3 \end{aligned}$$

$$\begin{aligned} x + y - z &= -8 \\ 2x - y + z &= -4 \\ -x + 2y + 2z &= 21 \end{aligned}$$

We can define them in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

And further define them in matrix notation

$$AX = C$$

- So, if we have a system of linear equations

$$AX = C$$

- We can algebraically reformat the equation by pre-multiplying both sides by the identity matrix

$$AX = C$$
$$A^{-1}X = A^{-1}C$$
$$IX = A^{-1}C$$
$$X = A^{-1}C$$

Solving simultaneous equations

- We can solve the following equation simultaneously through based algebraic manipulation

$$x + 5y = -8$$
$$-x - 2y = -1$$

solving for y, we reduce to

$$x + 5y + 8 = -x - 2y + 1$$

now add the two equations

$$x + 5y + 8$$
$$\underline{-x - 2y + 1}$$
$$3y = -9$$
$$y = -3$$

Solving simultaneous equations with matrices

- Or, we can use matrices

$$x + 5y = -8$$
$$-x - 2y = -1$$

$$\underbrace{\begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -8 \\ -1 \end{bmatrix}}_{C}$$

More complicated formulas

$$x + 5y = -8$$
$$-x - 2y = -1$$

$$A := \begin{pmatrix} 1 & 5 \\ -1 & -2 \end{pmatrix} \quad C := \begin{pmatrix} -8 \\ -1 \end{pmatrix}$$

$$\left(A^T \cdot A\right)^{-1} \cdot A^T \cdot C = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$x + y - z = -8$$
$$2x - y + z = -4$$
$$-x + 2y + 2z = 21$$

$$A := \begin{pmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ -1 & 2 & 2 \end{pmatrix} \quad C := \begin{pmatrix} -8 \\ -4 \\ 21 \end{pmatrix}$$

$$\left(A^T \cdot A\right)^{-1} \cdot A^T \cdot C = \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix}$$

Transformational View of Cartography

- Types of data/types of maps
- Map scale
- Dimensional
- Symbolization
- Generalization
- Data model

Transformations

- Goal is to express a transformation as an explicit math operation so that 0 → 1 in a fully described way and an inverse transformation 1 → 2 = 0
- Invertible transformations
  - Special subset, allows for prediction of error, spatial modeling
- Point-to-point transformations – very central, more than any other
- Can have multi-step transformations

© Arthur J. Lembo, Jr.
Salisbury University

Dimensional Transformations

- Coordinate transformations
  - Projections
- Geometric transformations
  - Measurements from coordinates
- Affine transformations
  - Rotation, translation, and scaling
- Statistical space transformations

Planar map transformations

- Distance between two points
  - Figured out 2,400 years ago by Pythagoras

$$d_{2to1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

  - Is this invertible? Only if we have three points
- Length of a line

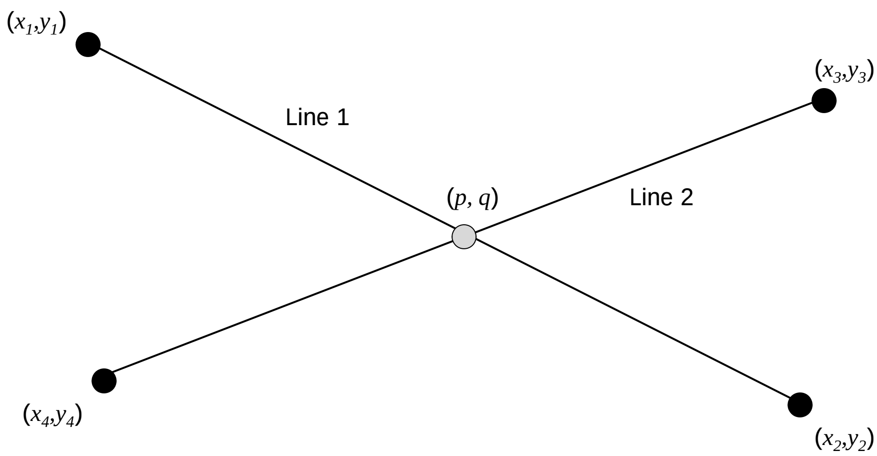
$$length = \sum_{i=1}^{npts-1} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Planar map transformations II

- Weighted average point
- Cities (x,y,P) P = population

$$\bar{x} = \frac{\sum_{i=1}^{npts} P_i x_i}{\sum_{i=1}^{npts} P_i} \qquad \bar{y} = \frac{\sum_{i=1}^{npts} P_i y_i}{\sum_{i=1}^{npts} P_i}$$

Intersection point of two lines



Intersection point of two lines

- If (x_1,y_1) and (x_2,y_2) lie on the same line then:

$$y_1 = a_1 + b_1 x_1$$

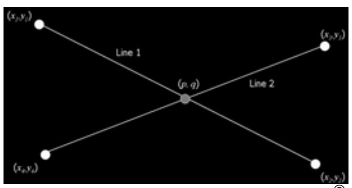
$$y_2 = a_1 + b_1 x_2$$

remember y = mx + b (the equation of a line)

- If (x_3,y_3) and (x_4,y_4) lie on the same line then:

$$y_3 = a_2 + b_2 x_3$$

$$y_4 = a_2 + b_2 x_4$$



Intersection point of two lines

- If an intersection exists, it must lie on both lines:

$$y = a_1 + b_1 x$$

$$y = a_2 + b_2 x$$

- Solve for simultaneous equations:

$$y - y = a_1 - a_2 + x(b_1 - b_2)$$

- Rearrange to get:

$$a_1 - a_2 = x(b_1 - b_2)$$

© Arthur J. Lembo, Jr.
Salisbury University

Intersection point of two lines

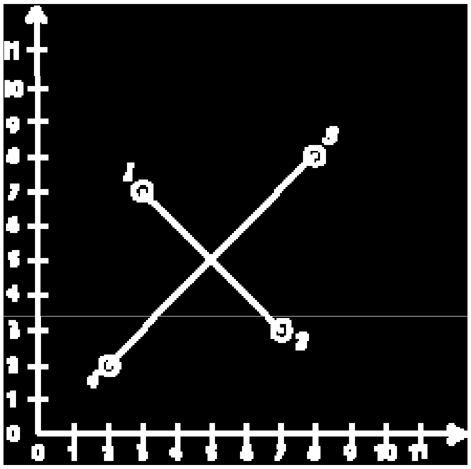
- Solving for x:

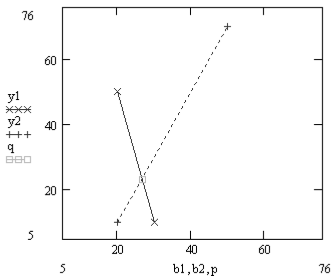
$$x = \frac{a_1 - a_2}{b_1 - b_2}$$

- And by substituting, solve for y:

$$y = a_1 + b_1 \frac{(a_1 - a_2)}{(b_1 - b_2)}$$

An example of line intersection





$$x_1 = 50$$
$$y_1 = 70$$
$$x_2 = 20$$
$$y_2 = 10$$

$$y_1 = a_1 + b_1 x_1$$
$$y_2 = a_1 + b_1 x_2$$
$$70 = a_1 + 50b$$
$$10 = a_1 + 20b$$
$$60 = 30b$$
$$b = 2$$
$$70 = a_1 + 40$$
$$a_1 = -30$$

$$y_3 = a_2 + b_2 x_3$$
$$y_4 = a_2 + b_2 x_4$$
$$10 = a_2 + 30b$$
$$50 = a_2 + 20b$$
$$-40 = 10b$$
$$b = -4$$
$$10 = a_2 + -120$$
$$a_2 = 130$$

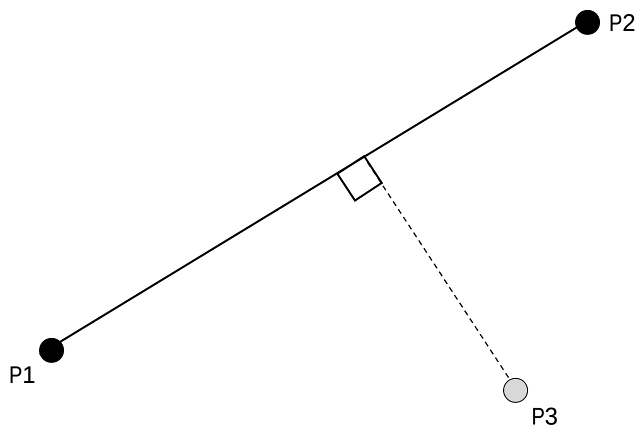
$$a_1 - a_2 = x(b_1 - b_2)$$
$$-30 - 130 = x(-4 - 2)$$
$$x = \frac{160}{6}$$
$$x = 26.667$$

$$y = a_1 + b_1 \frac{(a_1 - a_2)}{(b_1 - b_2)}$$
$$= -30 + {}^{-}2\left(\frac{{}^{-}30 - 130}{2 - {}^{-}4}\right)$$
$$y = 23.33$$

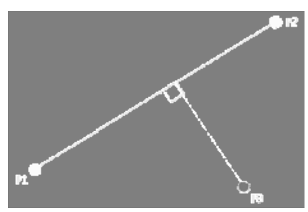
Intersection of two lines

- If the denominator above is 0, the lines are parallel
- If the denominator and the numerator above is 0, the lines are coincident
- In the computer, we have a real problem with 0
  - Actual zero almost never happens
  - Means we must check within limits of coordinate precision
- Solution is not elegant – test every combination of line segments
  - Bounding Box heuristic

Distance between a point and a line



Distance between a point and a line



- Equation for the line

$$P = \mathrm{P1} + u(\mathrm{P2} - \mathrm{P1})$$

- The shortest distance from P3 to P is a perpendicular line. This means the dot product of the line and the perpendicular is 0.

The dot product is often used to calculate the cosine of the angle between two vectors.

$$(\mathrm{P3} - \mathrm{P}) \text{ dot } (\mathrm{P2} - \mathrm{P1}) = 0$$

- Substitute for the equation of the line

$$[\mathrm{P3} - \mathrm{P1} - u(\mathrm{P2} - \mathrm{P1})] \text{ dot } (\mathrm{P2} - \mathrm{P1}) = 0$$

Distance between point and line

- Solve for *u*

$$u = \frac{(x3 - x1)(x2 - x1) + (y3 - y1)(y2 - y1)}{\left(\sqrt{(x2 - x1)^2 - (y2 - y1)^2}\right)^2}$$

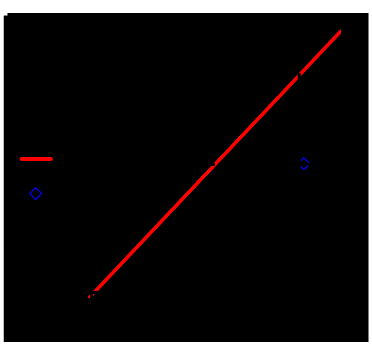
- Find the intersection point

$$x = x1 + u(x2 - x1)$$

$$y = y1 + u(y2 - y1)$$

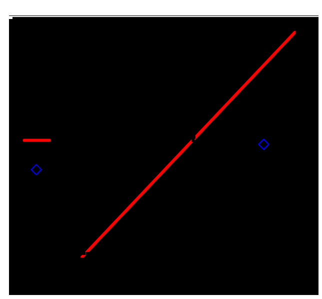
- Distance is length of line between P3 and intersection point

Clockwise vs. counter clockwise



$$(ptx - x1)\cdot(y2 - y1) - (x2 - x1)\cdot(pty - y1) = -20$$

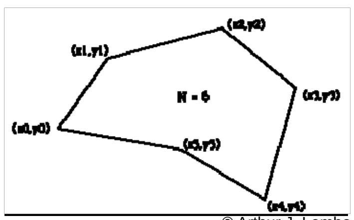
What if its on the line?

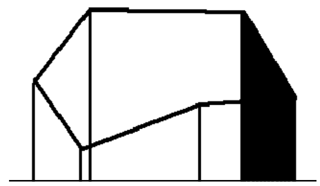


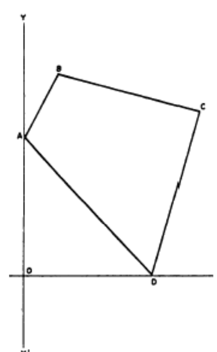
Area of a polygon

$$A = \frac{1}{2}\left|\sum_{i=1}^{npts+1}(x_i y_{i-1}) - (x_{i-1} y_i)\right|$$

- Project lines from each vertex to some outside perpendicular. Each area under the line is a triangle and a rectangle. Sum the areas as you move around the polygon. Outside areas get subtracted out







$$\frac{1}{2}\sum_{i=0}^{N-1}(x_i\, y_{i+1} - x_{i+1}\, y_i)$$

STATION	COORDINATES Y	X
A	591.64	0
B	847.60	125.66
C	694.07	716.31
D	0	523.62
A	591.64	0

591.64 × 125.66 = 74,345.48

847.60 × 716.31 = 607,144.36

694.07 × 523.62 = 363,428.93

0 × 0 = 0

Σ 1,044,918.76

STATION	COORDINATES Y	X
A	591.64	0
B	847.60	125.66
C	694.07	716.31
D	0	523.62
A	591.64	0

0 × 847.60 = 0

125.66 × 694.07 = 87,216.84

716.31 × 0 = 0

523.62 × 591.64 = 309794.53

Σ 397,011.37

What we have covered this semester

- Brief and fast review of GIS
  - Structure of GIS (HW/SW, file based, object based, object-relational) GIS data models (raster, vector, TIN)
- Object modeling and geodatabases
  - Geo-relational model, object oriented concepts, object behavior, benefits of geodatabases
  - Making features smart, qualities of features in a geodatabase, subtype, relationship classes
- Cartographic modeling
  - General principles in cartographic modeling, transfer of principles to Model Builder, spatial SQL
- Database design
  - Data models, entity-relationship diagrams, spatial entities in E-R diagrams, physical and logical design, design steps (model user view, define entities and relationships, transition to physical model)
  - ESRI Geodatabase structure (geodatabase, feature dataset, feature and object classes, relationship classes, rules)

What we have covered this semester

- Vector GIS operations
  - Vector functions (scale change, buffer, clip, overlay, Theissen polygons, Delaunay triangulation)
  - Map transformations (conformal, affine, polynomials, rubbersheeting), raster transformation (WTC example)
  - Spatial algorithms (concept of an algorithm, mathematics of analytical cartography: distance, intersection, containment)

How you got your hands dirty

- Review of classic GIS functions for site selection
- Getting real world data into GIS (even when the data isn't very good)
  - rubbersheeting and raster transformation
- Integration of classic GIS functions in a modern computing environment (model builder)
- Creating a geodatabase (controlled environment)
- Creating a geodatabase (non controlled environment)

What's on the written exam

- 10 definitions
- 3 short answers (lists, benefits/limitations, etc.)
- 1 computation (i.e. area, centroid, distance, intersection)
- 2 essays (conceptual ideas)

What's on the practicum

- Hands-on work
  - Create a geodatabase
    - Digitize boundaries
  - Create a small GIS analysis