Exploring Change through Activities, Graphs, and Mathematical Models

Presented by:

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Abstract

In this session we will discuss some ways to explore patterns of change through laboratory activities by investigating proportionality relationships and categorizing functions and mathematical models.

The activities illustrated are included in a workshop course intended for in-service middle school teachers. The course, titled "Mathematical Models and Modeling for Teachers," was developed under the auspices of a Salisbury University, NSF-funded, program identified as the Allied Delmarva Enhancement Program for Teachers (ADEPT). For additional information about this course see:
http://faculty.salisbury.edu/~dccathca/ModelsWorkshop/description.html
This paper can be found at:
http://faculty.salisbury.edu/~dccathca/DCC/NewOrleansNCTM.pdf

Preliminaries

Some Types of Mathematical Change

1) Suppose y is related to x by an explicit functional relationship $y = f(x)$.

We denote a change in x by Δx; the corresponding change in y, denoted by Δy is calculated by $\Delta y = f(x+ \Delta x) - f(x)$.

Assuming Δx is positive, the average rate of change in y with respect to x over the interval $[x, x+\Delta x]$ is $\Delta y/\Delta x$.

The percent change in y over the interval $[x, x+\Delta x]$ is $100[\Delta y/y]$.

2) Suppose x takes on integer values 0, 1, 2, … and the corresponding values of y are in the sequence $\{y_0, y_1, y_2, \ldots\}$.

   In this case, $\Delta x = 1$ and $\Delta y_n = y_{n+1} - y_n$.

   The percent change in y over the interval $[n, n+1]$ is $100[\Delta y_n/y_n]$

Some Proportionality Relationships

1) We say u is proportional to v, denoted by $u \propto v$, if for some nonzero constant k, $u = kv$.

2) We say u is inversely proportional to v, if $u \propto k/v$ for some nonzero constant k.

3) We say u is jointly proportional to v and w, denoted by $u \propto vw$, if for some nonzero constant k, $u = kuv$.

Two Examples

Δx	x	y	Δy	$\Delta y/\Delta x$
	0	3		
1	1	7	4	4
2	3	15	8	4
3	6	27	12	4

Δx	x	y	Δy	$\Delta y/\Delta x$	≈ % change
	0	2			
1	1	3	1	1	50
1	2	4.5	1.5	1.5	50
1	3	6.76	2.25	2.25	50

$y = 4x + 3$

$y_{n+1} - y_n = 3$

$\Delta y/\Delta x = 3$

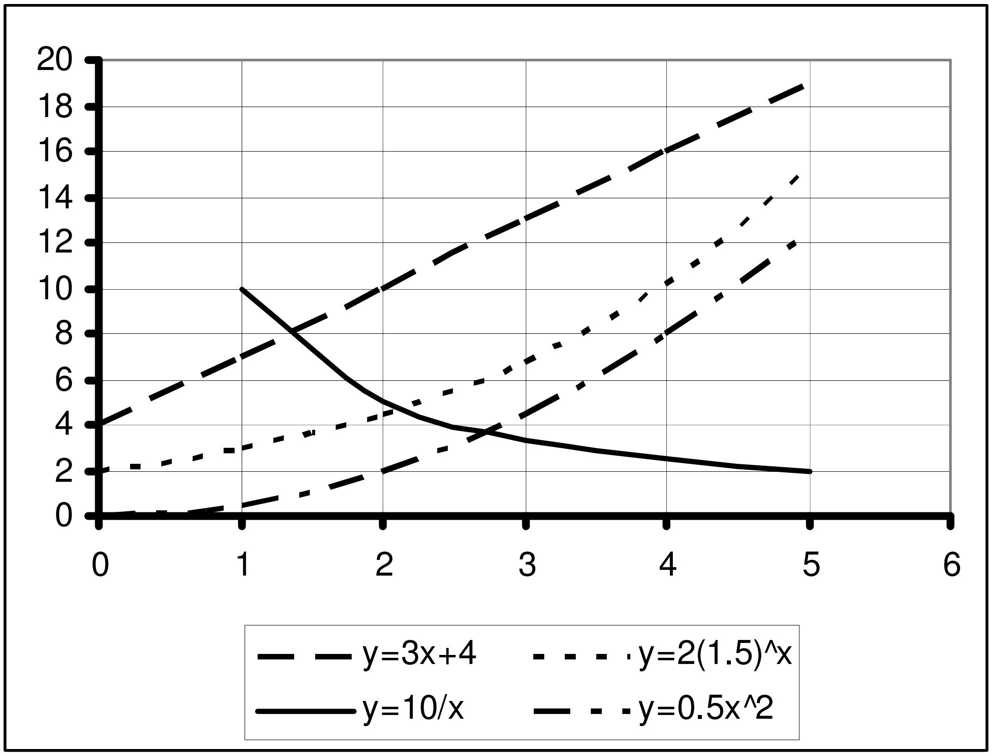
$\Delta y \propto \Delta x$

$y = 2(1.5)^x$

$y_{n+1} - y_n = (0.50)y_n$

$\Delta y_n \propto y_n$

Some Standard Functions and Their Graphs



Examples of Some Types of Growth

Growth Type	Sample Difference Equation	**Sample Functional** Equation
Arithmetic	$F_n = F_{n-1} + 2$, and $F_0 = 5$	$F_n = 2n + 5$
Quadratic	$G_n = G_{n-1} + 2n + 7$, and $G_0 = 5$	$G_0 = n^2 + 8n + 5$
Exponential	$H_t = H_{t-1} + 0.07H_{t-1}$, and $H_0 = 1000$	$H_t = 1000(1.07)^t$
Logistic	$P_t = P_{t-1} + 0.002P_{t-1}(2000 - P_{t-1})$, and $P_0 = 300$	

For arithmetic growth, the difference between successive terms is constant.

For quadratic growth, the difference between successive terms grows arithmetically and the difference between successive differences (second difference) is (are) constant.

For exponential growth, the difference between successive terms is proportional to the first of the two terms. The percent change is constant.

For logistic growth the difference between successive terms is jointly proportional to the first of the two terms and the difference between the limiting value and the first of the two terms.

Our Approach to Modeling from Data

Given a real world phenomenon to consider, we ask our students to demonstrate the following steps in fitting a model to data relative to the phenomenon:

- Pre Activity
  - o Formulate the key problem or question (show you understand the problem).
  - o Communicate preconceptions.
  - o Discuss the limitations, assumptions, and scope of the proposed model.
- Activity and Post Activity
  - o Collect and organize data.
  - o Analyze and interpret data.
  - o Fit an appropriate model. Vary parameters and test goodness of fit by an identified numerical criterion (sum of errors, average error, percent error).
  - o Discuss the limitations, assumptions, and scope of the model.
  - o Summarize and report findings. If possible, identify a proportionality relationship to validate your choice of models.
  - o Revisit preconceptions, reflect, describe, formulate, evaluate, support, generalize, research, and suggest.

Example 1: Weights and Springs

When weights are hung on a spring, the spring stretches. Describe the relationship between the mass hung on a spring and the amount the spring stretches

1. Take a few minutes and describe, in writing, what you think is the relationship between the mass hung on a spring and the amount the spring stretches.

2. Sketch a graph showing how you think a spring′s stretching is related to the mass hung on it. Place the mass of the weight hung on the spring on the horizontal axis and the anticipated stretch on the vertical axis. Simply provide a qualitative picture without concern for the numerical scales. (Label your axes.)

3. Describe how you will approach the task of determining the relationship between the stretching and the weight.

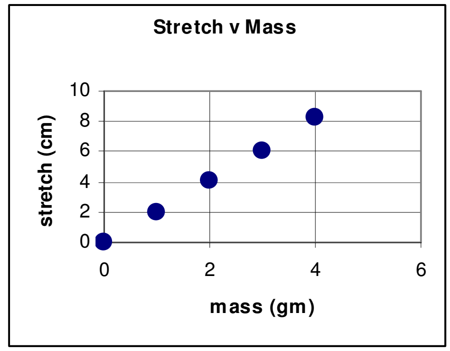
4. Carry out your task and complete the first two columns in the table below

Total Stretch of a Spring VS Mass Hung on Spring				
Mass (gm)	Total Stretch	Ave Rate of change	Model Prediction	Error
	(cm)	(cm/gm)	(cm)	(cm)

5. Sketch a graph of the relationship. Label your axes.

6. Develop a Model to Fit the Data. Comment on how well your model fits the data.

Total Stretch of a Spring VS Mass Hung on Spring				
Mass (gm)	Total Stretch	Ave Rate of change	Model Prediction	Error
	(cm)	(Δcm/Δgm)	(cm)	(cm)
0	0.0		0.0	0.0
1	2.0	2.0	2.1	0.1
2	4.1	2.1	4.2	0.1
3	6.1	2.0	6.3	0.2
4	8.2	2.1	8.4	0.2



Model: Total Stretch $\approx$ 2.1(mass hung on spring)
So, {stretch of spring} $\propto$ [mass hung on spring]

Example 2: Breaking Strength of Spaghetti Bridges

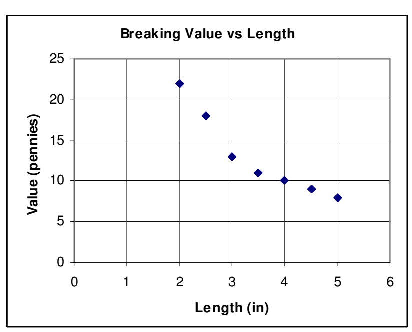
How is the length of a bridge related to the strength of the bridge?

1. To address this question we build spaghetti bridges and measure the strength of each bridge by the number of pennies required to break the spaghetti.

   - Identify issues, variables, and parameters.
   - Identify possible representations and preconceptions of the problem.
   - Refine the problem.
   - What physical principles affect the relationship?
   - What other factors or variables might come into play?
   - How can we represent these relationships? (Discuss not only the *type* of representation, but also the qualitative properties of the relationship.)

2. Collect data and complete the table below and sketch a graph of the relationship.

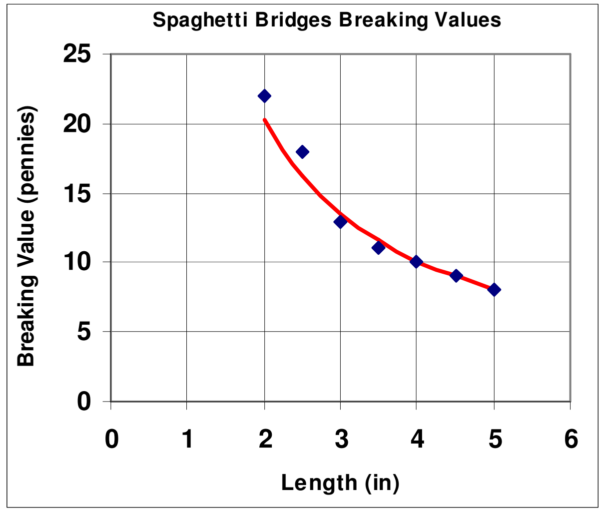
Length	Breaking Weight
2.0”	22
2.5”	18
3.0”	13
3.5”	11
4.0”	10
4.5”	9
5.0”	8



3. Experiment with the data and try to determine a relationship between the length of the spaghetti bridge and its breaking value.

Breaking Spaghetti Bridges

Bridge Length	Breaking Value		Model Value	Error
L	B	L x B	B = K / L	
(in)	(pennies)			
2.0	22	44.0	20	1.8
2.5	18	45.0	16	1.8
3.0	13	39.0	14	0.5
3.5	11	38.5	12	0.6
4.0	10	40.0	10	0.1
4.5	9	40.5	9	0.0
5.0	8	40.0	8	0.1



Model: $B \approx 40.5/L$
So, [breaking value] $\propto$ 1/[bridge length].
That is, the breaking value is inversely proportional to the bridge length.

The Draining Bottle

Explore the relationship between the height of water in a bottle and the rate at which the height of the water changes as the bottle is drained through a hole in the bottom. Think of water draining from a bathtub. How will the current height of the water affect the rate at which the water level is decreasing? Sketch a graph showing how you anticipate that the rate of change in height of the water will depend on the height of the water.

Equipment

- Clear two-liter soda bottle
- Nail
- Tape
- Ruler
- Watch with second hand
- Basins or bags to catch draining water (or go outside)

Procedure

- Punch a hole in the bottle with the nail about 5 centimeters from the bottom of the clear two-liter soda bottle.
- Tape the ruler vertically to the side of the bottle so that the 0 centimeter mark is aligned with the hole punched in the bottle.

- First person puts finger over the hole and fills bottle with water to a height of about 15 centimeters
- First person calls out as finger is removed from hole and calls out height of the water in whole centimeters as the water level passes that height
- Second person calls out elapsed time each time the level passes a height
- Third person records results in a table.

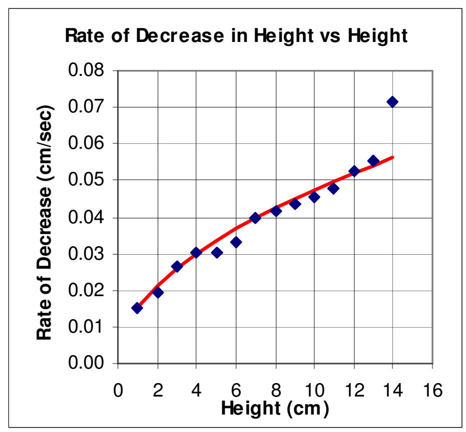
Elapsed Time t	Height of H_2O h	Δh	Δt	Δh/Δt

The rate at which the height is changing is Δh/Δt.

Any curve we fit to the data should have the property that when h is 0, Δh/Δt is also 0 (convince yourself by thinking about the water draining from the bottle.

We experiment with the data and try to determine a relationship between Δh/Δt and h.

Time (t) (sec)	Height (h) (cm)	Ave. Rate of Decrease in h **w.r.t** t (r in cm/sec)	Square Root **of** Height	Ratio of Rate to Sq Root	Model Value
0	15		3.87		
14	14	0.07	3.74	0.019	0.06
32	13	0.06	3.61	0.015	0.05
51	12	0.05	3.46	0.015	0.05
72	11	0.05	3.32	0.014	0.05
94	10	0.05	3.16	0.014	0.05
117	9	0.04	3.00	0.014	0.05
141	8	0.04	2.83	0.015	0.04
166	7	0.04	2.65	0.015	0.04
196	6	0.03	2.45	0.014	0.04
229	5	0.03	2.24	0.014	0.03
262	4	0.03	2.00	0.015	0.03
300	3	0.03	1.73	0.015	0.03
352	2	0.02	1.41	0.014	0.02
418	1	0.02	1.00	0.015	0.02



Model: Δh/Δt $\approx 0.015\sqrt{h}$

So, Δh/Δt $\propto \sqrt{h}$

Can we find a functional relationship between h and t?

A Brief Look a Three More Situations

Corn Growth

Plant four corn seeds in potting soil contained in a styrofoam cup. Place the cup where the plants will get exposure to the sun, keep the soil slightly moist, and keep track of the height (in cm) of any plants that develop. Measure your plants several times each week - maybe daily. Keep a diary of the life of your plants over a 21-28 day period beginning with the day you plant the seeds.

Before you actually plant the seeds, express your preconceptions concerning the plants' growth over time in the form of a graph. That is, sketch a graph showing how you think the plants' height will change over the next month or so.

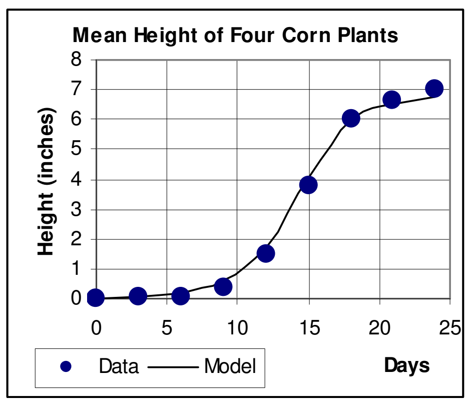
At the end of that 21-28 day period, put your data describing your plants' growth into a spreadsheet and produce some graphs. In particular, plot the average height of the plants (in cm) over the 21-28 day time interval. (The exact length of the time interval will be determined later.)

Describe any patterns you observe in the data or in your graphs. What can you say about the plants' growth rate over the time interval?

Try to fit a mathematical relationship to your data.

Tracking the Growth of Four Corn Plants (Heights in Inches)

Day	Plant A	Plant B	Plant C	Plant D	Mean Height	Model Predicts
0	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.25	0.00	0.00	0.06	0.06
6	0.00	0.25	0.00	0.00	0.06	0.19
9	0.00	1.00	0.25	0.25	0.38	0.60
12	1.00	2.00	2.00	1.00	1.50	1.75
15	3.00	4.25	5.00	3.00	3.81	4.07
18	5.00	6.00	6.00	7.00	6.00	5.99
21	6.25	7.00	6.00	7.25	6.63	6.53
24	6.50	7.50	6.00	8.00	7.00	6.78



Model: $H(t+3) - H(t) \approx 0.04H(t)(7.5 - H(t))^2$

So, the change in height over the next three days is jointly proportional to the current height and the square of the difference between the limiting height and the current height.

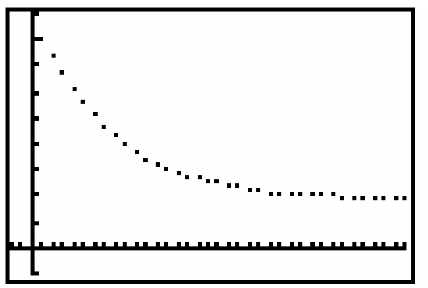
Cooling a Thermometer

A thermometer is placed in a cup of hot water until it records the temperature of the hot water, and then the thermometer is placed into a cup of room temperature water.

As in all activities, we communicate our preconceptions, both graphically and in writing, regarding the relationships being investigated. In this case, we comment on the rate at which we predict the probe will cooling, and on the relationship between the temperature recorded and the length of time the probe has been in room temperature water.

Will the rate of cooling be constant?

Here is a sample Temperature vs Time graph from a TI 82 LCD screen.



In this case the room temperature was 18.9^{o}C. To facilitate our analysis, we transfer the experimental data from the TI 82 to an Excel spreadsheet.

Experimenting with some “what if” scenarios can lead students to the conjecture that the average rate of change in the temperature of the probe, in degrees/sec, is proportional to the difference between the current probe temperature and the room temperature.

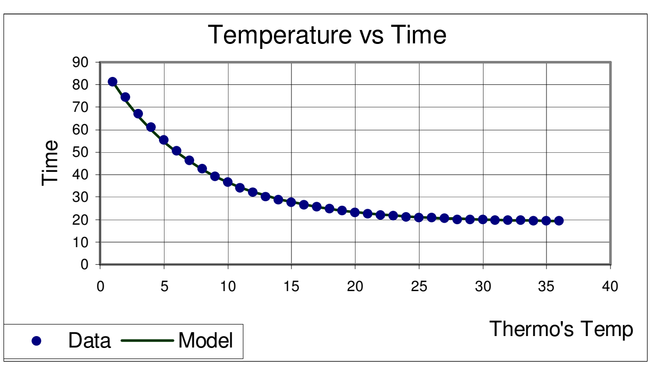
So, if we let P(n) = the probe’s temperature after n seconds, and we assume the room temperature is 18.9, then our relationship can be modeled by the difference equation

$P(1) = 81.1$

$P(n) – P(n-1) = k[P(n-1) – 18.9]$

So, the thermometer’s rate of cooling is proportional to the difference between its current temperature and the water temperature.

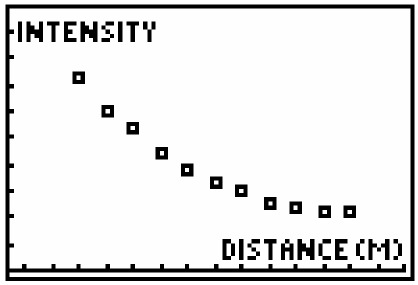
The graph below shows that when k = -0.13 the model above is a good fit to the data.



Light Intensity

Develop a mathematical model for the relationship between the intensity of a light source, in Watts per square meter, and distance from the source, in meters.

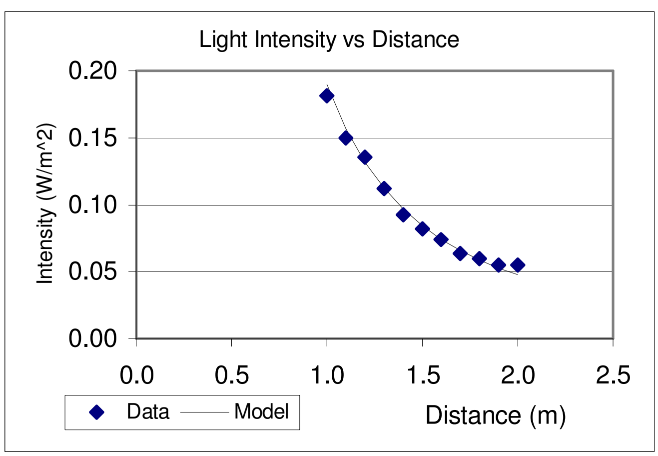
Here is a sample Intensity vs Distance graph from a TI 82 LCD screen.



As before, we transfer our experimental data from the TI 82 to an Excel spreadsheet. Doing so allows our students to explore "what if" scenarios. In doing our analysis we can conjecture that Intensity $\propto$ $1/(\text{Distance})^2$.

Model: $I \approx 0.19/s^2$, where I denotes the light's intensity and s denotes the distance from the light source.

We compare the model's predictions with the actual data in the graph below.



Additional Information

Web Site for the mathematical modeling workshop for middle school teachers:
http://faculty.salisbury.edu/~dccathca/ModelsWorkshop/description.html

Web sites for two other mathematical modeling courses:
- A course for prospective elementary school teachers- **http://faculty.salisbury.edu/~dccathca/MATH115/abstract.htm**
- A course for upper level mathematics majors- **http://faculty.salisbury.edu/~dccathca/MATH465/Syllabus.htm**