MATH 406 Session #23 Section 4.1 An Analytic Model of E^2

We are used to representing points in E^2 by ordered pairs of real numbers (x,y) and lines by equations of the form ax + by + c = 0. For the purposes of our work in the next chapters, we will associate points with ordered triples of the form (x,y,1). The two representations (x,y) and (x,y,1) of the points in E^2 are called *nonhomogeneous* and *homogeneous coordinates* respectively.

In this course, we will frequently denote points with column vectors $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

If a, b, and c are real numbers with at least one of them nonzero, then the set of all points (x,y,1) such that ax + by + c = 0 is a line. We will identify lines by row vectors $\begin{bmatrix} a & b & c \end{bmatrix}$.

The matrix equation $\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ is an abbreviation for $ax + by + c \cdot 1 = 0$.

Area of a Triangle with Known Vertices

In Class Session #3 we developed the following theorem.

Theorem 4.1.1 Given three noncollinear points in the plane $A(x_1,y_1,1)$, $B(x_2,y_2,1)$, $C(x_3,y_3,1)$, the area of the triangle determined by the three points is given by

$$A = \left(\frac{1}{2}\right) abs\left(\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}\right).$$

Recall how we evaluate determinants.

$$|a_{11}| = a_{11}. \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

There are other ways to evaluate determinants.

The Equation of a Line

If the points $A(x_1,y_1,1)$, $B(x_2,y_2,1)$, $C(x_3,y_3,1)$ are collinear and we tried to employ Theorem 4.1.1, what would we obtain for our area?

Suppose $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ are two given fixed points in the plane and $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ is any other point in the plane. The third (arbitrary) point is on the line through the first two (fixed) points if and only if

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

We find the equation of the line through $\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$.

$$\begin{vmatrix} x & 1 & 3 \\ y & 5 & 9 \\ 1 & 1 & 1 \end{vmatrix} = 0. \qquad -4x + 2y - 6 = 0 \qquad \text{or} \qquad \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0.$$

Intersection of Two lines

Given two lines l_1 and l_2 with equations

$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ respectively. Find their point of intersection if it exists.

If $a_1 b_2 - a_2 b_1 \neq 0$ then l_1 and l_2 intersect at the point $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ where

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$ *(Look up Cramer's Rule.)*