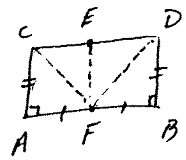
Ex 3.1: #1

Prove: The segment joining the midpoints of the summit and base of a Saccheri quadrilateral is perpendicular to both.

Proof Suppose $\square ABDC$ is a Saccheri quadrilateral with base $\overline{AB}$, summit $\overline{CD}$ and congruent sides $\overline{AC}$ and $\overline{BD}$ perpendicular to $\overline{AB}$.

Designate the unique midpoints of the base and summit F and E respectively.



Consider $\triangle CAF$ and $\triangle DBF$.

We have $\overline{AF} \cong \overline{FB}$, (F is midpoint of AB)

$\angle A \cong \angle B$, (Both are right angles)

and $\overline{AC} \cong \overline{BD}$ (Given)

So, $\triangle CAF \cong \triangle DBF$ by SAS.

(1) It follows that $\overline{CF} \cong \overline{DF}$ and $\angle CFA \cong \angle DFB$ because they are corresponding parts of congruent triangles.

Consider $\triangle CFE$ and $\triangle DFE$.

$\overline{CF} \cong \overline{DF}$ (Noted above)

$\overline{EF} \cong \overline{EF}$ (Reflexivity of $\cong$)

$CE \cong \overline{ED}$ (E is midpoint of $\overline{CD}$)

So $\triangle CFE \cong \triangle DFE$ by SSS.

(2) It follows that $\angle FEC \cong \angle FED$ and $\angle CFE \cong \angle DFE$, because they are corresponding parts of congruent triangles.

It follows directly from (2) above and the definition of perpendicular that $\overline{EF}$ is perpendicular to $\overline{CD}$ because $\angle FEC \cong \angle FED$. Of course $\overline{EF}$ is the segment joining the midpoints of the base and summit and $\overline{CD}$ is the summit. It remains to be shown that $\overline{EF}$ is perpendicular to the base $\overline{AB}$.

In order for us to prove the remaining part of this theorem we actually need an additional axiom that is frequently called the "Angle-Addition Postulate." (AAP)

AAP: If (a) D is in the interior of $\angle BAC$, (b) D' is in the interior of $\angle B'A'C'$ (c) $\angle BAD \cong \angle B'A'D'$, and (d) $\angle DAC \cong \angle D'A'C'$, then $\angle BAC \cong \angle BAD$.

In the statement of AAP if we associate A, B, C, D, E, F appropriately with A, B, C, D, A', B', C', D' it will follow that $\angle EFA \cong \angle EFB$.

Note that C is in the interior of $\angle EFA$ (a)
D is in the interior of $\angle EFB$ (b)
$\angle CFA \cong \angle DFB$ (c) by (1) above
$\angle CFE \cong \angle DFE$ (d) by (2) above

So we do have $\angle EFA \cong \angle EFB$. So, by the definition of perpendicular again $\overline{EF}$ is perpendicular to $\overline{AB}$. (We are done!)

Consequently, the segment joining the midpoints of the summit and base of a Saccheri quadrilateral is perpendicular to both.