3.2 #7

Suppose S is a unit circle in the Euclidean plane with center at O.

Prove: If A is a point in the interior of S and the Euclidean distance from A to O is $r<1$, then A's hyperbolic distance from O is given by $d(A,O) = \ln \frac{1+r}{1-r}$.

Proof #1

Suppose point A is in the interior of S and A's Euclidean distance from O is $r<1$. The definition on page 102 tells us that $d\rho = \frac{2\,dr}{1-r^2}$ where $\rho = d(A,O)$.

Integrating we obtain

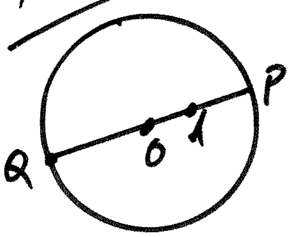
$$\rho = d(A,O) = \int_0^r \frac{2\,dt}{1-t^2} = 2\tanh^{-1} t \Big]_0^r = 2\left[\tfrac{1}{2}\ln\left|\frac{t+1}{t-1}\right|\right]_0^r$$

$$= \ln \frac{1+t}{1-t}\Big]_0^r = \ln\frac{1+r}{1-r} - \ln 1 = \ln\frac{1+r}{1-r}.$$

That is $d(A,O) = \ln \frac{1+r}{1-r}$.

Consequently, if A is a point in the interior of S and A is located Euclidean distance $r<1$ from the center O, then A's hyperbolic distance from O is $d(A,O) = \ln\frac{1+r}{1-r}$.

Proof #2 (Outline Only)



$$d(A,O) = \left|\ln \frac{\frac{OP}{OQ}}{\frac{AP}{AQ}}\right| = \left|\ln\frac{\frac{1}{1}}{\frac{1-r}{1+r}}\right| = \left|\ln\left|\frac{1+r}{1-r}\right|\right|$$

$$= \left|\ln\frac{1+r}{1-r}\right|$$