Details for solution for Section 4.5 #10. (See solution posted under Assignment #11 on web site.)

$\boxed{\begin{matrix}4.5\\ \#10\end{matrix}}$ Find any points invariant under $T\left(\begin{bmatrix}x\\y\\1\end{bmatrix}\right)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1\\ 0 & 0 & 1\end{pmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}$.

We seek $\begin{bmatrix}x\\y\\1\end{bmatrix}$ such that

(1) $\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}x\\y\\1\end{bmatrix}$, which is equivalent to (2) $\begin{matrix}-\frac{1}{2}x-\frac{\sqrt{3}}{2}y+\sqrt{3}=x\\ -\frac{\sqrt{3}}{2}x+\frac{1}{2}y+1=y\end{matrix}$

But (2) is equivalent to (3) $\begin{matrix}-\frac{3}{2}x-\frac{\sqrt{3}}{2}y=-\sqrt{3}\\ -\frac{\sqrt{3}}{2}x-\frac{1}{2}y=-1\end{matrix}$

(3) is equivalent to (4) $\begin{matrix}1x+\frac{\sqrt{3}}{3}y=\frac{2\sqrt{3}}{3}\\ 1x+\frac{\sqrt{3}}{3}y=\frac{2\sqrt{3}}{3}\end{matrix}$ and (4)

reduces to (5) $x=\frac{2\sqrt{3}}{3}-\frac{\sqrt{3}}{3}y$

So, all points on the line $1x+\frac{\sqrt{3}}{3}y-\frac{2\sqrt{3}}{3}=0$, that is the line $\begin{bmatrix}\sqrt{3} & 1 & -2\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=0$, are invariant under T. That line is therefore also invariant under T.

Check: Note the points we claim are invariant all have the form $\begin{bmatrix}\frac{2-t}{\sqrt{3}}\\ t\\ 1\end{bmatrix}$ for some $t\in\mathbb{R}$.

$$T\left(\begin{bmatrix}\frac{2-t}{\sqrt{3}}\\ t\\ 1\end{bmatrix}\right)=\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{2-t}{\sqrt{3}}\\ t\\ 1\end{bmatrix}=\begin{bmatrix}\frac{t-2}{2\sqrt{3}}-\frac{\sqrt{3}}{2}t+\sqrt{3}\\ \frac{t-2}{2}+\frac{1}{2}t+1\\ 1\end{bmatrix}=\begin{bmatrix}\frac{2-t}{\sqrt{3}}\\ t\\ 1\end{bmatrix} \checkmark$$