4.5 #10 continued

Find invariant lines under T. Note that $T^{-1} = T$;

So we seek $[a \ \ b \ \ c]$ such that

$$(6) \quad [a \ \ b \ \ c]\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} = k[a \ \ b \ \ c] \text{ for any } k \in \mathbb{R}.$$

(Theorem 4.2.4)

(6) is equivalent to (7) $\left[-\frac{1}{2}a + -\frac{\sqrt{3}}{2}b \quad -\frac{\sqrt{3}}{2}a + \frac{1}{2}b \quad \sqrt{3}a + b + c\right] = k[a\ b\ c]$

Setting $k=1$ we obtain

$$(8)\ \begin{cases} -\frac{1}{2}a - \frac{\sqrt{3}}{2}b = a \\ -\frac{\sqrt{3}}{2}a + \frac{1}{2}b = b \\ \sqrt{3}a + b + c = c \end{cases} \quad \text{that is} \quad (9)\ \begin{cases} -\frac{3}{2}a - \frac{\sqrt{3}}{2}b = 0 \\ -\frac{\sqrt{3}}{2}a - \frac{1}{2}b = 0 \\ \sqrt{3}a + b = 0 \end{cases}$$

(9) reduces to

$$(10)\ \begin{cases} a + \frac{\sqrt{3}}{3}b = 0 \\ a + \frac{\sqrt{3}}{3}b = 0 \\ a + \frac{\sqrt{3}}{3}b = 0 \end{cases} \quad \text{OR} \quad (11)\ a + \frac{\sqrt{3}}{3}b = 0$$

So, lines with equations of the form

$$\left(-\frac{\sqrt{3}}{3}t\right)x + (t)y + c = 0 \quad \text{for some } t, c \in \mathbb{R}$$

are invariant. That is all lines with slope $-\frac{\sqrt{3}}{3}$ are invariant. Equivalently lines of the form $[1 \ \ -\sqrt{3} \ \ c]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ are invariant for any $c \in \mathbb{R}$.

We also noted above that the line $[\sqrt{3} \ \ 1 \ \ -2]\begin{bmatrix} x \\ y \\ 1\end{bmatrix} = 0$ is invariant under T.