4.5 #10 Finding invariant lines by an alternate method

Note $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$

But since $T = T^{-1}$ we also have

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

So, $\begin{cases} -\frac{1}{2}x' - \frac{\sqrt{3}}{2}y' + \sqrt{3} = x \\ -\frac{\sqrt{3}}{2}x' + \frac{1}{2}y' + 1 = y \end{cases}$

Hence, the image of the line $\begin{bmatrix} A & B & C \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ is

$$A\left(-\frac{1}{2}x' - \frac{\sqrt{3}}{2}y' + \sqrt{3}\right) + B\left(-\frac{\sqrt{3}}{2}x' + \frac{1}{2}y' + 1\right) + C = 0$$

Rearranging terms

$$\left(-\frac{1}{2}A - \frac{\sqrt{3}}{2}B\right)x' + \left(-\frac{\sqrt{3}}{2}A + \frac{1}{2}B\right)y' + \left(\sqrt{3}A + B + C\right) = 0$$

The line and its image will be the same line if

$\begin{cases} -\frac{1}{2}A - \frac{\sqrt{3}}{2}B = A \\ -\frac{\sqrt{3}}{2}A + \frac{1}{2}B = B \\ \sqrt{3}A + B + C = C \end{cases}$ That is if $\begin{cases} -\frac{3}{2}A - \frac{\sqrt{3}}{2}B = 0 \\ -\frac{\sqrt{3}}{2}A - \frac{1}{2}B = 0 \\ \sqrt{3}A + B = 0 \end{cases}$

So, as in the previous method we note that the line $Ax + By + C = 0$ is invariant when $A = -\frac{1}{\sqrt{3}}B$ and $C \in \mathbb{R}$. Taking $B = -\sqrt{3}$ we have all lines of the form $1x - \sqrt{3}y + C = 0$ invariant under T.