$\boxed{\text{Ex 4.1 \#8}}$ Prove: Given two points $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ an equation for the line containing the two points is determined by the matrix equation

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

<u>Proof</u> Suppose $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ are two points and $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ represents an arbitrary point on the line through $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$. Theorem 4.1.1 tells us that if $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ is not on the line through $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ then a triangle is formed with area $\frac{1}{2}\text{abs}\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix}$. So, if $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ are collinear then no triangle is formed and $\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

The converse is also true. That is if $\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$, then $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ are collinear. Hence, the line through $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ is determined by the matrix equation

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$