Ex 4.1
#8 (version #2)

Prove: Given $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \in \mathbb{E}^2$, an equation for the line containing those points is determined by $\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$.

Proof Suppose $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \in \mathbb{E}^2$. The line containing those two points has equation (in point-slope form):

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \quad \text{or} \quad (y-y_1) = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$$

Moving to standard form

$$y = \left(\frac{y_2-y_1}{x_2-x_1}\right)x - \left(\frac{y_2-y_1}{x_2-x_1}\right)x_1 + y_1$$

$$(x_2-x_1)y = (y_2-y_1)x - (y_2-y_1)x_1 + (x_2-x_1)y_1$$

$$(y_2-y_1)x - (x_2-x_1)y + \left[(x_2-x_1)y_1 - (y_2-y_1)x_1\right] = 0$$

$$-\begin{vmatrix} y_1 & y_2 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix} y - \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = 0 \quad \text{(defn. determinant)}$$

$$\begin{vmatrix} y_1 & y_2 \\ 1 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \quad \text{(defn. determinant)}$$

Thus, the line containing the points $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$ is determined by $\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$.