Euclidean Geometry – A synthetic approach

Undefined terms (objects, relations): point, line, plane, on (through, contains, incident with), between (order), congruent

Axioms of Connection (Incidence):

I1: For any two distinct points, there is exactly one line incident with both of them.
I2: There exists at least two distinct points incident with each line.
I3: There exists at least three points incident with each plane, not all three of which are incident with the same line.
I4: For any three distinct points that are not all incident with the same line, there is exactly one plane incident with all three of those points.
I5: If two distinct points that are incident with a line are also incident with a plane, then each point incident with the line is also incident with the plane.
I6: There exists at least four points, not all of which are incident with the same line.
I7: If two distinct planes are incident with a point, then they are both incident with exactly one line.

Axioms of Betweeness (Order):

O1: For any two distinct points A and B incident with a line, there exists at least one other point C on the line which is between them. We denote this situation by writing A-C-B or equivalently B-C-A.
O2: For any three distinct points incident with a line, exactly one of the three is between the other two.
O3: Every segment has exactly one midpoint.
O4: If all the points incident with a line and a triangle are contained entirely in the Same plane and if the line is incident with a side of the triangle at a point other than a vertex, then the line is also incident with one of the other sides of the triangle. (Pasch's Postulate)

Dedekind's Axiom: If all points incident with a line are partitioned into two nonempty and disjoint sets L and R in such a way that every point of L is to the left of every point of R, then there is exactly one point T which is the boundary of the partitioning and T is either the rightmost point of L or the leftmost point of R.

Axiom of Separation:

S1: Each line partitions each plane which contains it into two sets called half-planes called the left half-plane and the right half-plane in such a way that:

1) Exactly one of the following is true: every point of the plane is in the left half-plane or in the right half-plane or on the line.
2) For any point A in one half-plane and any point B in the other half-plane, the line segment containing A and B is incident with a point on the line which formed the two half-planes.
3) If two points of a line are in the same half-plane, then every point between them is in the same half-plane.

Axioms of Congruence:

C1: If $\overline{AB}$ is contained in line *l* and *C* is a point distinct from A and B and C is either on *l* or another line *m,* then there exists on each side of C on any line containing it, *l* or *m*, exactly on point D such that $\overline{AB}$ is congruent to $\overline{CD}$.

C2: Each segment is congruent to itself. If any (first) segment is congruent to a second segment, the second segment is congruent to the first. (Congruence is reflexive and symmetric.)

C3: If a first segment is congruent to a second segment and the second is congruent to a third segment, then the first segment is congruent to the third. (Congruence is transitive.)

C4: If *l* is any line and P is any point, then there is exactly one line through P that is perpendicular to *l*.

C5: Suppose ∠ ABC is any angle, if O is a point on some line *l*, and M is a second distinct point on *l*, then on either side of *l* there is exactly one ray with its endpoint at O which forms an angle with ray OM which is congruent to ∠ ABC.

C6: Each angle is congruent to itself. If a first angle is congruent to a second angle, the second angle is congruent to the first.

C7: If a first angle is congruent to a second angle and the second is congruent to a third angle, then the first angle is congruent to the third.

C8: If two sides and the included angle of one triangle are congruent respectively to two corresponding sides and included angle of another triangle, then the triangles are congruent. ***(Side-Angle-Side (SAS) Axiom)***

Axioms of Trichotomy:

T1: For any two segments a first and a second, the first is either shorter than, congruent to, or longer than the second.

T2: For any two angles a first and a second, the first is either smaller than, congruent to, or larger than the second.

Axiom of Parallelism (Playfair):

P1: If *l* is any line and P is any point which is not on *l*, then there is exactly one other line *m* in the same plane with *l* and P which contains P and does not have any points in common with *l*.