Hemmer's Axioms for Synthetic Euclidean Geometry
(Geometry without Distance)

Undefined terms (objects, relations): point, line, plane, on (through, contains, incident with), between, congruent

Axioms of Connection (Incidence):

I1: For any two distinct points, there is exactly one line incident with both of them.
I2: There exists at least two distinct points incident with each line.
I3: There exists at least three points incident with each plane, not all three of which are incident with the same line.
I4: For any three distinct points that are not all incident with the same line, there is exactly one plane incident with all three of those points.
I5: If two distinct points that are incident with a line are also incident with a plane, then each point incident with the line is also incident with the plane.
I6: There exists at least four points, not all of which are incident with the same line.
I7: If two distinct planes are incident with a point, then they are both incident with exactly one line.

Theorem 1: Two distinct lines cannot both be incident with more than one point.

Theorem 2: If two distinct lines intersect, then they are both incident with exactly one plane.

Axioms of Order:

O1: For any two distinct points A and B incident with a line, there exists at least one other point C on the line which is between them. We denote this situation by writing A-C-B or equivalently B-C-A.
O2: For any three distinct points incident with a line, exactly one of the three is between the other two.

Defn: Given two distinct points A and B incident with a line, ***segment*** AB consists of points A and B and all points of the line between A and B. Points A and B are called the ***endpoints*** of the segment. A ***midpoint*** of the segment AB is a point M such that A-M-B and segment AM is congruent to segment MB.

O3: Every segment has exactly one midpoint.

Defn: Points P, Q, and R all incident with the same line are called ***collinear***; Points P and Q not incident with the same line are called ***non-collinear***.

Lemma 1: Three non-collinear points A, B, and C determine three distinct segments segment AB, segment AC, and segment BC.

Defn: The three segments of the Lemma 1 comprise at ***triangle***. The three segments are called the ***sides*** of the triangle. Each of the three points is called a ***vertex***. Each vertex is incident with exactly two sides. The side that does not contain a vertex is said to be

opposite that vertex. The triangle is named by the letters at its vertices. In this case triangle ABC denoted by ΔABC.

O4: If all the points incident with a line and a triangle are contained entirely in the Same plane and if the line is incident with a side of the triangle at a point other than a vertex, then the line is also incident with one of the other sides of the triangle.

What do we mean by each of the following: A line contains a point; a line contains a segment; a plane contains a line; a plane contains a segment; a plane contains a triangle; a point is contained in a line; two lines intersect; two plane intersect; a line and a plane intersect; two lines are parallel; two planes are parallel; a line and a plane are parallel; two lines are skew; etc?

Lemma 2: There exists an infinite number of points incident with each line.

Dedekind's Axiom: If all points incident with a line are partitioned into two nonempty and disjoint sets L and R in such a way that every point of L is to the left of every point of R, then there is exactly one point T which is the boundary of the partitioning and T is either the rightmost point of L or the leftmost point of R.

Axiom of Separation:

S1: Each line partitions each plane which contains it into two sets called half-planes called the left half-plane and the right half-plane in such a way that:

1) Exactly one of the following is true: every point of the plane is in the left half-plane or in the right half-plane or on the line.
2) For any point A in one half-plane and any point B in the other half-plane, the line containing A and B is incident with a point on the line which formed the two half-planes.
3) If two points of a line are in the same half-plane, then every point between them is in the same half-plane.

Define each of the following: ray, angle, vertex of an angle, interior of an angle, exterior of an angle, included angle.

Axioms of Congruence:

C1: If segment AB is contained in line l and C is a point distinct from A and B and C is either on l or another line m, then there exists on each side of C on any line containing it, l or m, exactly on point D such that AB is congruent to CD.

C2: Each segment is congruent to itself. If any (first) segment is congruent to a second segment, the second segment is congruent to the first. (Congruence is reflexive and symmetric.)

C3: If a first segment is congruent to a second segment and the second is congruent to a third segment, then the first segment is congruent to the third. (Congruence is transitive.)

Define: perpendicular, right angle

C4: If *l* is any line and P is any point, then there is exactly one line through P that is perpendicular to *l*.

C5: Suppose angle ABC is any angle, if O is a point on some line *l*, and M is a second distinct point on *l*, then on either side of *l* there is exactly one ray with its endpoint at O which forms an angle with ray OM which is congruent to angle ABC.

C6: Each angle is congruent to itself. If a first angle is congruent to a second angle, the second angle is congruent to the first.

C7: If a first angle is congruent to a second angle and the second is congruent to a third angle, then the first angle is congruent to the third.

C8: If two sides and the included angle of one triangle are congruent respectively to two corresponding sides and included angle of another triangle, then the triangles are congruent. ***(Side-Angle-Side (SAS) Axiom)***

Define: (a) What does it mean to say one segment is longer than another?
(b) What does it mean to say one angle is larger than another?

Axioms of Trichotomy:

T1: For any two segments a first and a second, the first is either shorter than, congruent to, or longer than the second.

T2: For any two angles a first and a second, the first is either smaller than, congruent to, or larger than the second.

Theorem 3: Given two triangles ABC and XYZ if segment AB is congruent to Segment XY, angle B is congruent to angle Y, and angle A is congruent to angle X then triangles ABC and XYZ are congruent.

Theorem 4: If three sides of one triangle are congruent respectively to three corresponding sides of another triangle, then the triangles are congruent.

Axiom of Parallelism (Playfair):

P1: If *l* is any line and P is any point which is not on *l*, then there is exactly one other line *m* in the same plane with *l* and P which contains P and does not have any points in common with *l*.

Given two lines intersected by a third, define: interior angles, alternate-interior pairs.

Define: exterior angle of a triangle, straight angle, vertical angles, supplementary angles, adjacent angles.

Theorem 5: If two angles are each supplementary to a third angle, then the two given angles are congruent.

Theorem 6: If two lines intersect, the vertical angles so formed are congruent.

Theorem 7: An exterior angle of a triangle is greater than either of the two interior angles that are not adjacent to it.

Theorem 8: If l_1 and l_2 are two lines in the same plane and if they are both intersected by some third line l_3 and if the alternate interior angles so formed are congruent, then l_1 and l_2 are parallel.

Theorem 9: If l_1 and l_2 are two distinct parallel lines and they are both intersected by a third line l_3, then the alternate interior angles formed must be congruent.

Define what it means to say two angles comprise a third angle. Define what it means to say three angles comprise a fourth angle.

Theorem 10: The angles of any triangle comprise one straight angle.