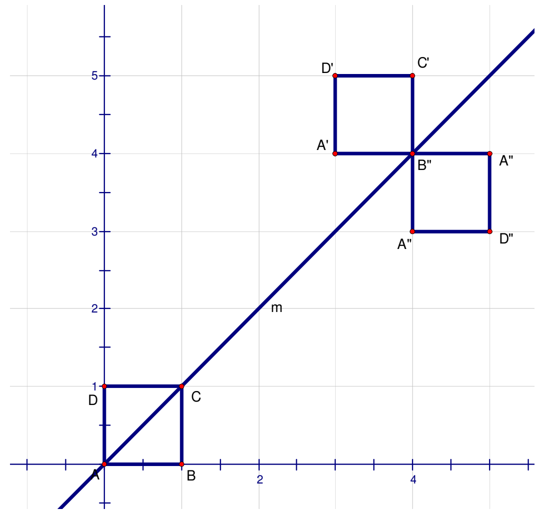
MATH 406 Sample Exercises

Example 1:

Suppose $T_1\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

If $T_3 = T_2 \circ T_1$, then $T_3\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.



Are any points invariant under T_3?

Are there any points such that $T_3\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$?

$T_3\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ iff (1) $\left\{\begin{array}{l} 0x + 1y + 4 = x \\ 1x + 0y + 3 = y \\ 0x + 0y + 1 = 1 \end{array}\right\}$.

(1) is equivalent to (2) $\begin{array}{l} -1x + 1y = -4 \\ 1x + -1y = -3 \end{array}$. (2) is equivalent to (3) $\begin{array}{l} -1x + 1y = -4 \\ 0x + 0y = -7 \end{array}$.

System (3) is inconsistent. So, there can be no invariant points under T_3.

Are any lines invariant under T_3?

Method 1:

We note that $T_3^{-1}\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

Suppose $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$. It follows that $\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. That is $\begin{array}{l} x = y' - 3 \\ y = x' - 4 \end{array}$.

The image of the line Ax + By + C = 0 under T_3 becomes A(y'-3) + B(x'- 4) + C = 0. After simplification we see that the image of Ax + By + C = 0 becomes Bx' + Ay' + (C – 3A – 4B) = 0. So, the line Ax + By + C = 0 is invariant iff A = B and 3A + 4B = 0. That last condition implies that each of A, B, C must be zero. But we cannot have each of A, B, C zero and have an equation for a line, so there can be no invariant lines under T_3.

Method 2: In this method we appeal to Theorem 4.2.4 and seek $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ such that

(4) $$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix} = k\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$ for any k $\in$ R.

(4) is equivalent to $\begin{bmatrix} u_2 & u_1 & -3u_1 - 4u_2 + u_3 \end{bmatrix} = k\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$.

Setting k = 1 we obtain $u_1 = u_2$ and $-3u_1 = 4u_2$. So, $u_1 = u_2 = 0$ and $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ cannot represent a line. Hence, there are no invariant lines under T_3.

Example 2:

Find all invariant lines under T defined by $T\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{2-\sqrt{3}}{2} & \frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{2+\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$.

Note the matrix for T^{-1} is $\begin{bmatrix} \frac{2+\sqrt{3}}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{2-\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, $\begin{bmatrix} \frac{2+\sqrt{3}}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{2-\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} \frac{2+\sqrt{3}}{2}x' - \frac{3}{2}y' \\ \frac{1}{2}x' + \frac{2-\sqrt{3}}{2}y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. Therefore, the image of the line l: Ax + By + C = 0 is

given by $A(\frac{2+\sqrt{3}}{2}x' - \frac{3}{2}y') + B(\frac{1}{2}x' + \frac{2-\sqrt{3}}{2}y') + C = 0$. Or, after simplification, by $(\frac{2+\sqrt{3}}{2}A + \frac{1}{2}B)x' + (-\frac{3}{2}A + \frac{2-\sqrt{3}}{2}B)y' + C = 0$. The line l will be invariant iff $\frac{\sqrt{3}}{2}A + \frac{1}{2}B = 0$ and $-\frac{3}{2}A - \frac{\sqrt{3}}{2}B = 0$. That is line l will be invariant iff $A = -\frac{\sqrt{3}}{3}B$. So, lines of the form $\begin{bmatrix} -\sqrt{3} & 3 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ for any $c \in$ R are invariant under T. So, all lines with slope $\frac{\sqrt{3}}{3}$ are invariant under T.

(See Example 4.6.5 in the textbook.)