LP Problems & Duality

Given an $m \times n$ matrix $\mathbf{A}$, an m-vector $\mathbf{b}$, and an n-vector $\mathbf{c}$, the *standard maximum problem* determined by $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$, denoted by SMP[$\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$] is the problem

$$(*) \qquad \text{maximize} \quad \mathbf{x} \cdot \mathbf{c} \qquad \text{subject to} \quad \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{Ax} \leq \mathbf{b}.$$

The *feasible set* for this SMP is $\{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{Ax} \leq \mathbf{b}\}$. The function $\mathbf{x} \cdot \mathbf{c}$ is called the *objective function* for the problem.

The *dual* of the SMP[$\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$] is the problem

$$(**) \qquad \text{minimize} \quad \mathbf{y} \cdot \mathbf{b} \qquad \text{subject to} \quad \mathbf{y} \geq \mathbf{0} \text{ and } \mathbf{A}^T \mathbf{y} \geq \mathbf{c}.$$

This dual problem (**) is an example of a standard minimum problem. This standard minimum problem may be denoted by smp[$\mathbf{A}^T$, $\mathbf{c}$, $\mathbf{b}$]. The *feasible set* for this smp is $\{\mathbf{y} \in \mathbf{R}^m : \mathbf{y} \geq \mathbf{0} \text{ and } \mathbf{A}^T\mathbf{y} \geq \mathbf{c}\}$. The function $\mathbf{y} \cdot \mathbf{b}$ is called the *objective function* for the problem.

Example: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

a. Formulate the SMP[$\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$] and its dual smp[$\mathbf{A}^T$, $\mathbf{c}$, $\mathbf{b}$].

b. Solve both the primal SMP and its dual by a graphing method.
c. Solve the SMP problem using a computer.