The Discrete Logistic Growth Model

Suppose that M is the maximum size of a population that can be sustained with specific resources and that the relative growth rate is proportional to the difference between M and the current size of the population.

Suppose we denote the size of the population at the beginning of each time interval [t_k, t_{k+1}] , k = 0, 1, 2, … by x_k. For this model we assume that the relative growth rate over the interval (t_k, t_{k+1}] is proportional to (M – x_k). That is

(1) $$\frac{x_{k+1} - x_k}{x_k} = r(M - x_k)$$ **for some nonzero *r*.**

Note that $\frac{x_{k+1} - x_k}{x_k} = rM(1 - \frac{x_k}{M})$ **and substituting *k* for *rM* we have the following form equivalent to (1).**

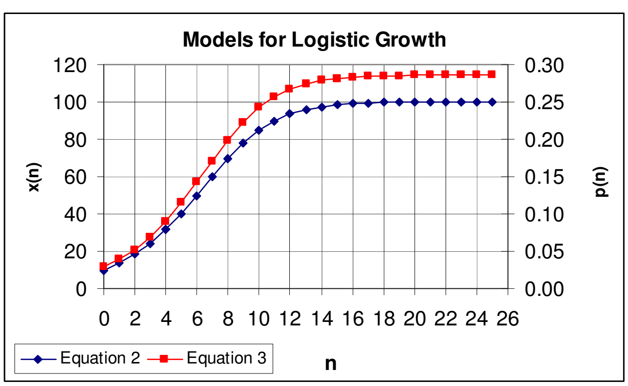
(2) $$\frac{x_{k+1} - x_k}{x_k} = k(1 - \frac{x_k}{M}).$$

A very clever change of variables, $x_n = [(k+1)/k]Mp_n$ and $b = k + 1$, allows us to transform equation (2) into the form

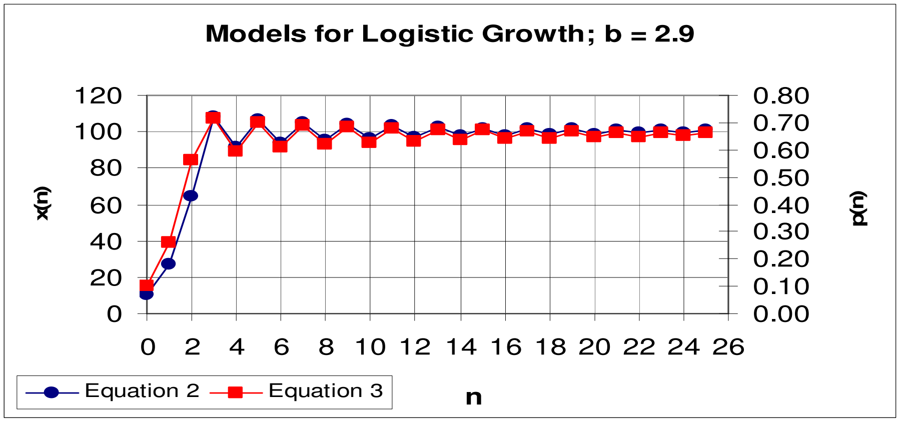
(3) $$p_{n+1} = bp_n(1 - p_n).$$ (Exercise 4)

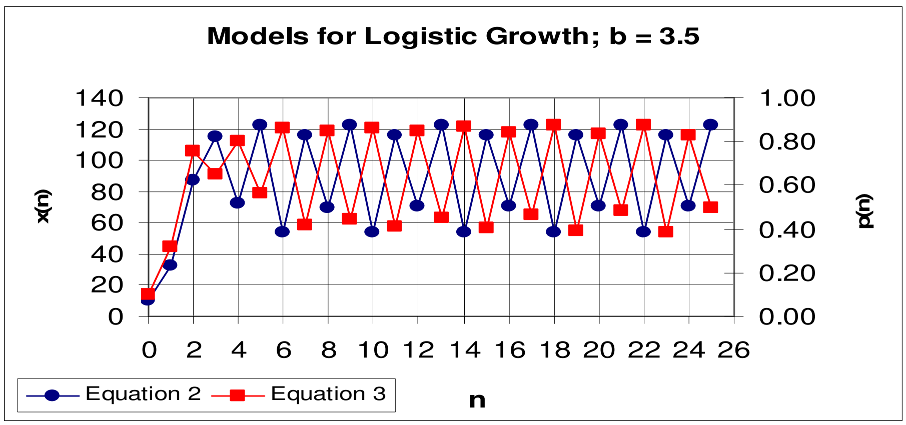
What do we gain and what do we loose as we move from considering equation (1) to considering equation (3)?

Graphs for equations (2) and (3) where x_0 = 10, M = 100, and k = 0.4 are shown below. Here p_0 = 0.029 and b = 1.4.



The solutions of equation (2) and (3) depend on x_0, *k*, *M* and x_0 and *b* respectively. In the graphs below we see that the equations display quite different behavior for different values of the parameters.





The Continuous Logistic Growth Model

M is again the maximum size of a population that can be sustained and the relative growth rate is $k(1 - x/M)$ where x is the current size of the population. The predicted population size x is given by the solution for the differential equation

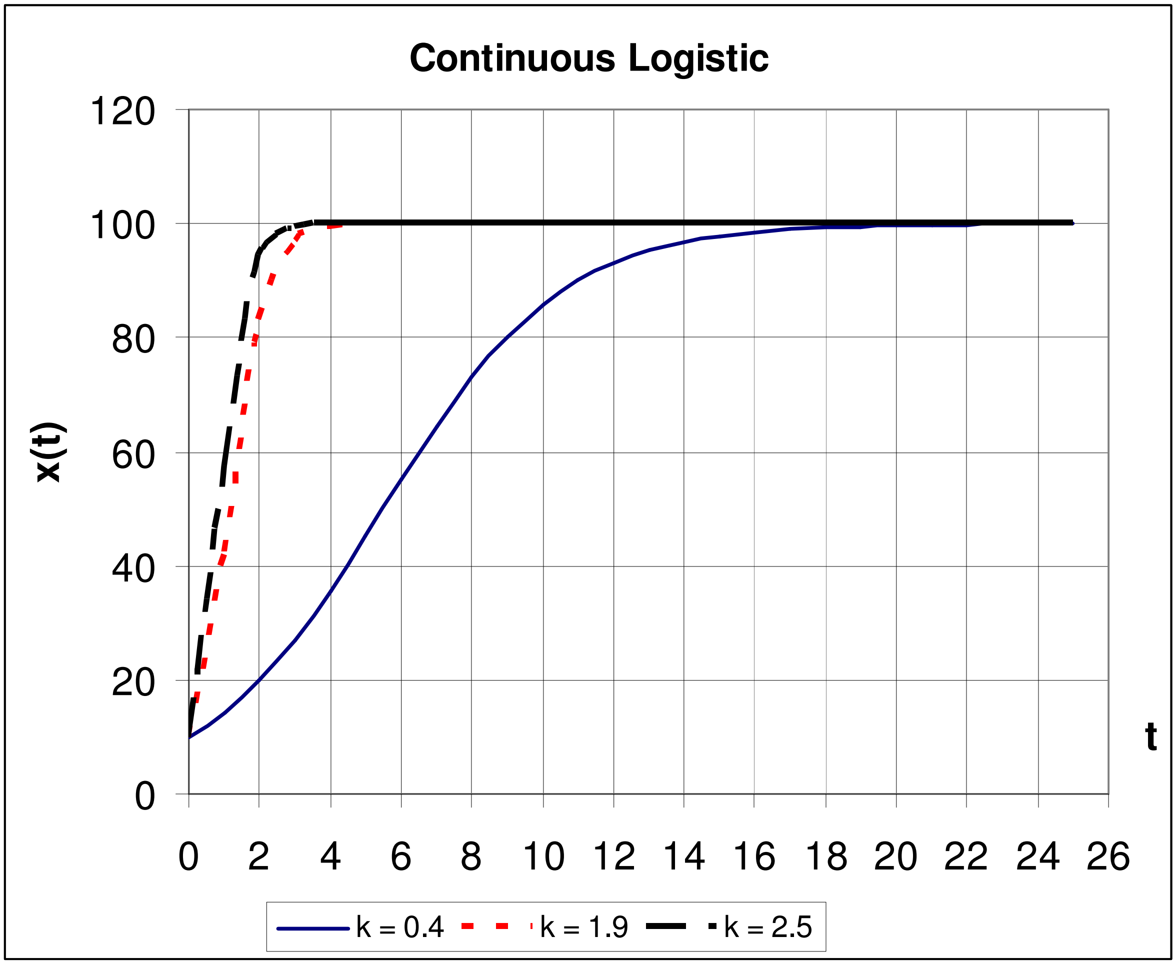
$$\textbf{(4)} \qquad \frac{1}{x}\frac{dx}{dt} = k\left(1-\frac{x}{M}\right),\ t \geq 0,\ x(0) = x_0$$

The solution can be shown to be

$$\textbf{(5)} \qquad x(t) = \frac{x_0 M}{x_0 + (M - x_0)e^{(-kt)}},\ t \geq 0 \qquad \text{(Exercise 3.)}$$

Since the solution in the continuous case is monotonic increasing, the predictions based on the discrete model differ significantly from those based on the continuous model.

The graphs below display the behavior of the continuous model for some different values of the parameter k. In each case $x_0 = 10$ and $M = 100$.



Returning to our discussion of the discrete logistic model for notational simplicity we follow the lead of Maki and Thompson and revert to using lower case letters x for population size and write equation (3) as

(6) $x_{n+1} = bx_n(1 - x_n)$, **n = 0, 1, 2, …**

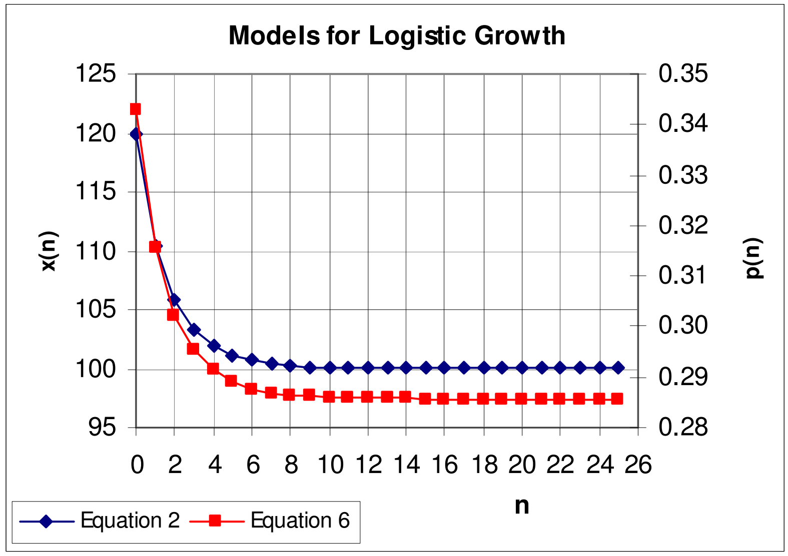
Note that in equation (6) x has a different meaning than in equation (2). (Explain)

We now consider the effect of different values of x_0 and b on the behavior of the solutions of equation (6).

What will occur if x_0 is greater than M in equation (2)?

x(0) =	120	x(0)=	0.343
M =	100.0	b=	1.400
k =	0.4		

n	x(n)	n	x(n)
0	120.0	0	0.343
1	110.4	1	0.315
2	105.8	2	0.302
3	103.3	3	0.295
4	102.0	4	0.291
5	101.2	5	0.289
6	100.7	6	0.288
7	100.4	7	0.287
8	100.2	8	0.286
9	100.1	9	0.286
10	100.1	10	0.286
11	100.1	11	0.286
12	100.0	12	0.286
13	100.0	13	0.286
14	100.0	14	0.286
15	100.0	15	0.286
16	100.0	16	0.286
17	100.0	17	0.286
18	100.0	18	0.286
19	100.0	19	0.286
20	100.0	20	0.286
21	100.0	21	0.286
22	100.0	22	0.286
23	100.0	23	0.286
24	100.0	24	0.286
25	100.0	25	0.286



Consider a few more parameter changes.

x(0)=	0.2	0.2	0.2
b =	3.50	2.00	0.50
	A	B	C
n	x(n)	x(n)	x(n)
0	0.200	0.200	0.200
1	0.560	0.320	0.080
2	0.862	0.435	0.037
3	0.415	0.492	0.018
4	0.850	0.500	0.009
5	0.446	0.500	0.004
6	0.865	0.500	0.002
7	0.409	0.500	0.001
8	0.846	0.500	0.001
9	0.456	0.500	0.000
10	0.868	0.500	0.000
11	0.400	0.500	0.000
12	0.840	0.500	0.000
13	0.470	0.500	0.000
14	0.872	0.500	0.000
15	0.391	0.500	0.000
16	0.833	0.500	0.000
17	0.486	0.500	0.000
18	0.874	0.500	0.000
19	0.385	0.500	0.000
20	0.828	0.500	0.000
21	0.497	0.500	0.000
22	0.875	0.500	0.000
23	0.383	0.500	0.000
24	0.827	0.500	0.000