Markov Chain Processes (MCP)

We consider a system that can be one of N possible *states* $S = \{s_1, s_2, \ldots, s_N\}$ and we observe the system at n successive times. If the system is in state s_i at the k^{th} observation and in state s_j at the $(k+1)^{th}$ observation, we say the process has made a transition from s_i to s_j at the k^{th} *observation*, *trial* , *step*, or *stage* of the process.

Let p_{ij} be the conditional probability that a system in s_i at the k^{th} observation is in state s_j at the $(k+1)^{th}$ observation, $i, j = 1, 2, \ldots, N$. These probabilities are *transition probabilities*. A process (system) is a *Markov chain (MCP)* if the p_{ij}'s depend only on i and j, the states occupied on the k^{th} and $(k+1)^{th}$ observations.

The $N \times N$ matrix $\mathbf{P} = (p_{ij})$ is the *transition matrix* for the MCP. Each row of **P** is a probability vector.

Let $p_{ij}(m)$ be the conditional probability that a system in s_i initially is in s_j on the m^{th} observation.

We denote the $N \times N$ matrix $(p_{ij}(m))$ by $\mathbf{P}(m)$.

$\mathbf{P}(m) = \mathbf{P}^m$

Let **P** be the transition matrix for a MCP, if there is an integer $r \geq 1$ such the $\mathbf{P}^r$ has only positive entries, then the Markov chain is *regular*.

Exercise 1: Suppose that the matrices $\mathbf{P}_1$, P_2, and $\mathbf{P}_3$ are transition matrices for MCP's. Which, if any, of those processes are regular?

$$\mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0.4 & 0.6 \end{bmatrix}; \mathbf{P}_2 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.2 & 0.5 & 0.3 \\ 0.5 & 0.5 & 0 \end{bmatrix}; \mathbf{P}_3 = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

Let **P** be the transition matrix for a regular MCP, then there is a unique probability vector **s** with positive coordinates such that

$$\mathbf{sP} = \mathbf{s} \text{ and moreover } lim_{n\to\infty}\mathbf{P}^n = \begin{bmatrix} s \\ \vdots \\ s \end{bmatrix}.$$

The coordinates of the vector **s** give the *stable probabilities* for the process and is also known as the *stationary vector* for the matrix **P**.

A MCP is called *ergodic* if for every pair of states s_i and s_j there is an integer m, which depends on i and j such that $p_{ij}(m) > 0$.

Exercise 2: Which, if any, of the MCP's in *Exercise 1* are ergodic?

The i^{th} state of a MCP is called *absorbing* if $p_{ii} = 1$.

A MCP is *absorbing* if it has at least one absorbing state and transition from each nonabsorbing state to some absorbing state is (eventually) possible.

Exercise 3: Which, if any, of the MCP's in *Exercise 1* are absorbing?

Consider the MCP with matrix $\mathbf{P}_1$ of *Exercise 1.* We calculate a few powers of **P**.

$$\mathbf{P}_1^{2} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}; \quad \mathbf{P}_1^{3} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}; \quad \mathbf{P}_1^{4} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}; \ldots$$

$$\mathbf{P}_1^{10} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}; \ldots, \mathbf{P}_1^{20} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}.$$

Suppose we partition the matrix $\mathbf{P}_1$ **as follows**

$\mathbf{P}_1 = \begin{bmatrix} I_1 & O \\ R & Q \end{bmatrix}$ where $\mathbf{I}_1$ is the *1 x 1* identity matrix, **O** is the *1 x 1* zero matrix, **R** is the *2 x 1* matrix $\begin{bmatrix} 0.3 \\ 0 \end{bmatrix}$, and **Q** is the *2 x 2* matrix $\begin{bmatrix} 0.7 & 0 \\ 0.4 & 0.6 \end{bmatrix}$. Now we calculate powers of $\mathbf{P}_1$ in terms of the sub-matrices $\mathbf{I}_1$, **O, R,** and **Q.**

$$\mathbf{P}_1^{2} = \begin{bmatrix} I_1 & O \\ R & Q \end{bmatrix}\begin{bmatrix} I_1 & O \\ R & Q \end{bmatrix} = \begin{bmatrix} I_1 & O \\ R+QR & Q^2 \end{bmatrix}$$

$$\mathbf{P}_1^{3} = \begin{bmatrix} I_1 & O \\ R & Q \end{bmatrix}\begin{bmatrix} I_1 & O \\ R+QR & Q^2 \end{bmatrix} = \begin{bmatrix} I_1 & O \\ R+QR+Q^2R & Q^3 \end{bmatrix}$$

:

$$\mathbf{P}_1^{k} = \begin{bmatrix} I_1 & O \\ R+QR+Q^2R+\ldots+Q^{k-1}R & Q^k \end{bmatrix}$$

Can we make a conjecture about $lim_{k\to\infty}\mathbf{Q}^k$ and $lim_{k\to\infty}\mathbf{P}_1^{k}$?

Exercise 4: For and k = 1, 2, 3, … let $\mathbf{R}_k = \mathbf{R} + \mathbf{QR} + \mathbf{Q}^2\mathbf{R} + \ldots + \mathbf{Q}^{k-1}\mathbf{R}$.

a. Show that $(\mathbf{I}_2 - \mathbf{Q})(\mathbf{I}_2 + \mathbf{Q} + \mathbf{Q}^2 + \ldots + \mathbf{Q}^{k-1}) = \mathbf{I}_2 - \mathbf{Q}^k$.

b. Can we establish that $(\mathbf{I}_2 - \mathbf{Q})^{-1}$ is $lim_{k\to\infty} (\mathbf{I}_2 + \mathbf{Q} + \mathbf{Q}^2 + \ldots + \mathbf{Q}^k)$?

c. Can we establish that $(\mathbf{I}_2 - \mathbf{Q})^{-1}\mathbf{R}$ is $lim_{k\to\infty} \mathbf{R}_k$.

d. If we let $\mathbf{N} = (\mathbf{I}_2 - \mathbf{Q})^{-1}$, can we show that $lim_{k\to\infty}\mathbf{P}_1^{k} = \begin{bmatrix} I_1 & O \\ NR & O \end{bmatrix}$?

The matrix **N** is called the *fundamental matrix* for the MCP with matrix $\mathbf{P}_1$.