Some Notes on Series

(I) Geometric Series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rS_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$1S_n - rS_n = a - ar^n$$

$$(1-r)\,S_n = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

Suppose $0 < r < 1$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

Example 1

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

$$S_\infty = \lim_{n\to\infty} S_n = \frac{1}{1-\frac{1}{2}} = 2$$

Example 1a

$$S_\infty = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}$$

$$S_\infty = 1$$

Example 1c $S_\infty = r + r^2 + r^3 + \cdots + \frac{1}{r^n} + \cdots$

$$S_\infty = \frac{r}{1-r}$$

(II) Not a Geometric Series

Example 2 for $(0 < r < 1)$

$$S_\infty = \sum_{k=1}^{\infty} k r^k = 1r + 2r^2 + 3r^3 + \cdots$$

$$\begin{array}{llllll}
1r + & & & & & \\
1r^2 + 1r^2 + & & & & & \\
1r^3 + 1r^3 + 1r^3 + & & & & & \\
1r^4 + 1r^4 + 1r^4 + 1r^4 + \cdots & & & & & \\
\vdots \quad \vdots \quad \vdots \quad \vdots & & & & & \\
1r^n + 1r^n + 1r^n + 1r^n + \cdots + 1r^n + \cdots & & & & & \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \quad \vdots & & & & &
\end{array}$$

$$S_\infty = \frac{r}{1-r} + \frac{r^2}{1-r} + \frac{r^3}{1-r} + \frac{r^4}{1-r} + \cdots + \frac{r^n}{1-r} + \cdots$$

$$S_\infty = \frac{1}{1-r} \cdot \frac{r}{1-r} = \frac{r}{(1-r)^2}$$ (Factor $\frac{1}{1-r}$ out of above expression)

$$S_\infty = \sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2}$$

Example 3

$$S_\infty = \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

Example 4

$$S_\infty = 2\left[1\left(\tfrac{1}{2}\right)^2\right] + 3\left[1\left(\tfrac{1}{2}\right)^3\right] + 4\left[2\left(\tfrac{1}{2}\right)^4\right]$$
$$+ 5\left[3\left(\tfrac{1}{2}\right)^5\right] + 6\left[5\left(\tfrac{1}{2}\right)^6\right]$$
$$+ 7\left[8\left(\tfrac{1}{2}\right)^7\right] + 8\left[13\left(\tfrac{1}{2}\right)^8\right]$$
$$+ 9\left[21\left(\tfrac{1}{2}\right)^9\right] + 10\left[34\left(\tfrac{1}{2}\right)^{10}\right] + \cdots$$

$$S_\infty = \sum_{K=2}^{\infty} K\, F_{K-1} \left(\frac{1}{2}\right)^K$$

$(K-1)^{st}$ Fibonacci number

$$S_\infty = \sum_{k=1}^{\infty} (k+1) F_k \left(\frac{1}{2}\right)^{k+1}$$
$$= \frac{1}{2}\left[\sum_{K=1}^{\infty} (K+1) F_K \left(\frac{1}{2}\right)^K\right]$$
$$= \frac{1}{2}\sum_{K=1}^{\infty} K F_K \left(\frac{1}{2}\right)^K + \frac{1}{2}\sum_{K=1}^{\infty} F_K \left(\frac{1}{2}\right)^K$$
$$= 6$$

(Details left as exercise)