Consider the difference equation (2.13) given on page 42 in our text.

$$(2.13) \qquad x_{n+1} = b x_n (1 - x_n), \quad n = 0, 1, 2, \ldots$$

where $0 < x_0 < 1$ and $0 < b \le 4$.

We define a function f by $f(x) = bx(1-x)$ for $0 \le x \le 1$. It follows that

$$(2.14) \qquad x_{n+1} = f(x_n), \quad x_0 \text{ given}, \ 0 \le x_0 \le 1, \ n = 0, 1, 2, \ldots$$

$$f'(x) = b - 2bx$$

$f'(x) = 0$ if $x = \frac{1}{2}$; $f'(x) > 0$ if $x < \frac{1}{2}$; $f'(x) < 0$ if $x < \frac{1}{2}$.

So, $f(\frac{1}{2}) = \frac{b}{4}$ is the maximum value of f on $0 \le x \le 1$.

Hence f maps $[0,1]$ onto itself. (Recall $0 < b \le 4$)

Suppose $x_0 = 0.1$ and $b = 2$. Here $f(x) = 2x(1-x)$

$$x_1 = f(x_0) = f(0.1) = 0.18$$
$$x_2 = f(x_1) = f(0.18) = 0.2952$$
$$x_3 = f(x_2) = f(0.2952) \approx 0.41611$$
$$\vdots$$

Can we find a fixed point for f in the interval $(0,1)$? That is, is there x^*, $0 < x^* < 1$ for which $x^* = f(x^*)$? Let's see

$x^* = f(x^*)$ if $\quad x^* = 2x^*(1-x^*)$

$$x^* = 2x^* - 2(x^*)^2$$

$$2(x^*)^2 - 2x^* + x^* = 0$$

$$2(x^*)^2 - x^* = 0$$

$$x^*(2x^* - 1) = 0$$

$$x^* = 0 \text{ or } x^* = \tfrac{1}{2}$$

In general for $f(x) = bx(1-x)$ we seek x^* such that

$$x^* = bx^*(1-x^*)$$

$$0 = (b-1)x^* - b(x^*)^2$$

$$x^*(bx^* - (b-1)) = 0$$

$$x^* = 0 \text{ or } x^* = \frac{b-1}{b}.$$