Some Discrete Models for Growth Processes

Let $T = \{t_k\}$ be a set of equally spaced times for $k = 0, 1, 2, \dots$
x_k denote the size of the population at time t_k

Some notation

$\Delta t_k = t_{k+1} - t_k$ (length of a time interval)

$\Delta x_k = x_{k+1} - x_k$ (change in the size of the population over the time interval $(t_k, t_{k+1}]$)

Model 1 (Exponential Growth) [P. 37]

The size of Δx_k is proportional to x_k.
That is, for some $r \neq 0$

$$\Delta x_k = x_{k+1} - x_k = r x_k, \quad k = 0, 1, 2, \dots$$

or

$$x_{k+1} = x_k + r x_k = (1+r) x_k, \quad k = 0, 1, 2, \dots$$

We have shown that in general

$$x_k = (1+r)^k x_0 \quad \text{for } k = 0, 1, 2, \dots$$

Model 2 (Logistic Growth) [P. 42]

The size of Δx_k is jointly proportional to x_k and $(M - x_k)$ where M is the maximum size of the population.
That is, for some $c > 0$

$$\Delta x_k = x_{k+1} - x_k = c x_k (M - x_k), \quad k = 0, 1, 2, \dots$$

$$x_{k+1} = x_k + c x_k (M - x_k), \quad k = 0, 1, 2, \dots$$

or

$$x_{k+1} = x_k + \hat{c} x_k \left(1 - \frac{x_k}{M}\right), \quad k = 0, 1, 2, \dots \text{ for some } \hat{c} > 0.$$

Equivalently

$$\frac{x_{k+1} - x_k}{x_k} = \hat{c}\left(1 - \frac{x_k}{M}\right) \text{ where } \hat{c} \text{ is a constant.}$$

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Equivalently $\dfrac{x_{k+1} - x_k}{x_k} = \hat{c}\left(1 - \dfrac{x_k}{M}\right)$ where $\hat{c}$ is a constant.