Introduction to Markov Chain Processes (MCP)

Example

An electric power company checks its main generator once each quarter to forestall blackouts due to equipment failure. For simplicity, we assume there are two possible outcomes for each quarterly inspection: W, the generator is in good working order and needs no repair; D, the generator is defective and needs repair. If the outcome in one quarter is D, repairs will be made and it is extremely likely that the next quarter's outcome will be W. If the outcome from one quarter is W, no repairs are made and there is a fair chance that the next quarter's outcome will be D.

Let us assume that if a given quarter's outcome is W that the probability that the next quarter's outcome is W is 0.6 and the probability that the next quarter's outcome is D is 0.4. Let us also assume that if a given quarter's outcome is D the probability that the next quarter's outcome is W is 0.9 and the probability that the next quarter's outcome is D is 0.1.

What kinds of mathematical representations might we employ to communicate the information in the previous paragraph?

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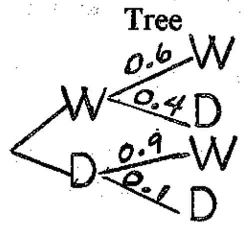
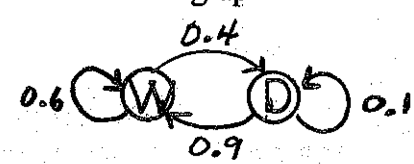
Matrix

Current Outcome	Next Outcome	
	W	D
W	0.6	0.4
D	0.9	0.1

Tree



Digraph



We refer to the quarterly inspections as ***trials*** in a process. Each trial is associated with an ***outcome*** from the set {W, D}. The particular outcome will be called the ***state*** of the system in that period. The probabilities indicating the likelihood of moving from one state to another (or the same) state in a given period are called ***transition probabilities***.

In our example, if we let $W = s_1$ and $D = s_2$ denote the states for this process, then we let p_{ij} = the probability of making a transition from s_i to s_j in the next period. The matrix P defined by

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.9 & 0.1 \end{bmatrix}$$

is the transition matrix for the process discussed in this example.

We will consider the long-term behavior of the inspection process. In general, we will want to know the probability that the process will be in a particular state in period n.

Let $\pi_i(n)$ = probability the process will be in s_i in period n. $\pi_i(n)$ is referred to as a ***state probability***. We will be interested in a probability vector $\Pi(n) = [\pi_1(n) \quad \pi_2(n)]$.

Given the state probabilities for a particular period we can determine the state probabilities for the next period using matrix multiplication.

$$\Pi(n + 1) = \Pi(n)P$$

Also,

$$\Pi(n + 1) = \Pi(n)P = [\Pi(n-1)P]P = \Pi(n - 1)P^2 = \ldots = \Pi(0)P^{n+1}$$

Let's look at the state probabilities for future quarterly periods for our electric power company's generator.

state prob	0	1	2	3	4	5	6	7	8	9
$\pi_1(n)$	1	0.6	0.72	0.684	0.6948	0.69156	0.69253	0.69224	0.692327	0.692301
$\pi_2(n)$	0	0.4	0.28	0.361	0.3052	0.30844	0.30746	0.307759	0.307672	0.307698

As we continue the process we find that the probability of being in a particular state after a large number of periods is independent of the initial state of the system. The probabilities we approach after a large number of transitions are referred to as the ***steady-state probabilities.*** As we look at the table above we can see that as n gets larger,

$$[\pi_1(n + 1) \quad \pi_2(n + 1)] \approx [\pi_1(n) \quad \pi_2(n)].$$

So, if we seek the steady-state probabilities in this example, we are looking for p_1 and p_2 such that p_1 and p_2 are non-negative, $p_1 + p_2 = 1$, and

$$[p_1 \;\; p_2] = [p_1 \;\; p_2]\begin{bmatrix} 0.6 & 0.4 \\ 0.9 & 0.1 \end{bmatrix} = [0.6p_1 + 0.9p_2 \quad 0.4p_1 + 0.1p_2]$$

So, $p_1 = 0.6p_1 + 0.9p_2$, $p_2 = 0.4p_1 + 0.1p_2$, and $p_1 + p_2 = 1$.

We solve the following linear system,

$-0.4p_1 + 0.9p_2 = 0$
$0.4p_1 - 0.9p_2 = 0$
$1.0p_1 + 1.0p_2 = 1$

and obtain $p_1 = \dfrac{9}{13} \approx 0.692307$ and $p_2 = \dfrac{4}{13} \approx 0.307692$.

So, in the long run subsequent to an inspection the process will be in state W about 69% of the time and in state D about 31% of the time.

When we consider a system that can be in one of N possible states $\{s_1, \ldots, s_N\}$ on successive observations, if the system is in s_i on the k^{th} observation on s_j on the $(k+1)^{th}$ observation we say the system has made a transition from s_i to s_j at the k^{th} ***trial, step,*** or ***stage*** of the process. We let p_{ij} denote the conditional transition probability that the process will make a transition from s_i to s_j at the k^{th} trial. The process is a ***Markov chain process (MCP)*** if the transition probabilities p_{ij} depend only on s_i and s_j and not on the number, k, of the trial.