MCP (Review)

probability vector $\bar{p} = [p_1, \cdots, p_n]$, $p_i \geq 0$, $\sum p_i = 1$

probability matrix $\mathbb{P}_{n\times n} = (p_{ij})_{n\times n}$, rows probability vectors

$\bar{p}$, probability vector
$\mathbb{P}$, probability matrix $\Rightarrow \bar{p}\,\mathbb{P}$ a probability vector

$\mathbb{P}, \mathbb{Q}$ probability matrices $\Rightarrow \mathbb{P}\mathbb{Q}$ a probability matrix

$\mathbb{P}$ a probability matrix $\Rightarrow \exists\, \bar{t} \neq \bar{0} \ni \bar{t}\,\mathbb{P} = \bar{t}$.

$\mathbb{P}$ a transition matrix for a MCP

$$\mathbb{P}^m = (p_{ij}^{(m)})$$ where $p_{ij}^{(m)}$ is the m-step transition probability of moving from s_i to s_j.

Initial probability distribution $\bar{p}^{(0)} = (p_1^{(0)}, \cdots, p_n^{(0)})$

m^{th} step probability distribution $\bar{p}^{(m)} = (p_1^{(m)}, \cdots, p_n^{(m)})$

$$\bar{p}^{(m)} = \bar{p}^{(0)}\,\mathbb{P}^m$$

$\mathbb{P}$ regular if $\mathbb{P}^m > \mathbb{O}$ for some m. (Regular MCP)

$\mathbb{P}$ regular $\Rightarrow \exists!\, \bar{t}$ (probability vector) $\ni \bar{t}\,\mathbb{P} = \bar{t}$

$$\lim_{m\to\infty} \mathbb{P}^m = \begin{bmatrix} \bar{t} \\ \vdots \\ \bar{t} \end{bmatrix}, \quad \lim_{m\to\infty} \bar{p}^{(0)}\,\mathbb{P}^m = \bar{t} \quad \forall\, \bar{p}^{(0)}.$$

Absorbing MCP

s_i absorbing state if $p_{ii} = 1$.

s_j transient state if $p_{jj} < 1$

suppose $s_1, \dots, s_k$ are absorbing states and $s_{k+1}, \dots, s_n$ are transient states.

$$\mathbb{P} = \left[\begin{array}{ccc|ccc} p_{11} & \cdots & p_{1k} & p_{1,k+1} & \cdots & p_{1n} \\ \vdots & & & \vdots & & \vdots \\ p_{k1} & \cdots & p_{kk} & p_{k,k+1} & \cdots & p_{kn} \\ \hline p_{k+1,1} & \cdots & p_{k+1,k} & p_{k+1,k+1} & \cdots & p_{k+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n1} & \cdots & p_{nk} & p_{n,k+1} & & p_{nn} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{I}_{k\times k} & \mathbb{O}_{k\times n-k} \\ \hline \mathbb{R}_{n-k,k} & \mathbb{Q}_{n-k,n-k} \end{array}\right]$$

$$\mathbb{P}^m = \left[\begin{array}{c|c} \mathbb{I} & \mathbb{O} \\ \hline \mathbb{R}_m & \mathbb{Q}^m \end{array}\right] \to \left[\begin{array}{c|c} \mathbb{I} & \mathbb{O} \\ \hline \mathbb{N}\mathbb{R} & \mathbb{O} \end{array}\right]$$

as $m \to \infty$

where $\mathbb{N} = (\mathbb{I} - \mathbb{Q})^{-1}$. (fundamental matrix)

Note,

$$(\mathbb{I} - \mathbb{Q})^{-1} = \mathbb{I} + \mathbb{Q} + \mathbb{Q}^2 + \cdots + \mathbb{Q}^k + \cdots$$

Consider the absorbing MCP

$$\mathbb{P} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{array}\right) \end{array} = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbb{O}_{2\times 3} \\ \hline R_{3\times 2} & \mathbb{Q}_{3\times 3} \end{array}\right)$$

1st non-absorbing state is s_3
2nd non-absorbing state is s_4
3rd non-absorbing state is s_5

$$\mathbb{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

We denote the (i,j)-entry of $\mathbb{Q}^2$ by $q_{ij}{}^{(2)}$.

Note

(1) $$q_{ij}{}^{(2)} = \sum_{k=1}^{3} q_{ik}q_{kj} = q_{i1}q_{1j} + q_{i2}q_{2j} + q_{i3}q_{3j}$$

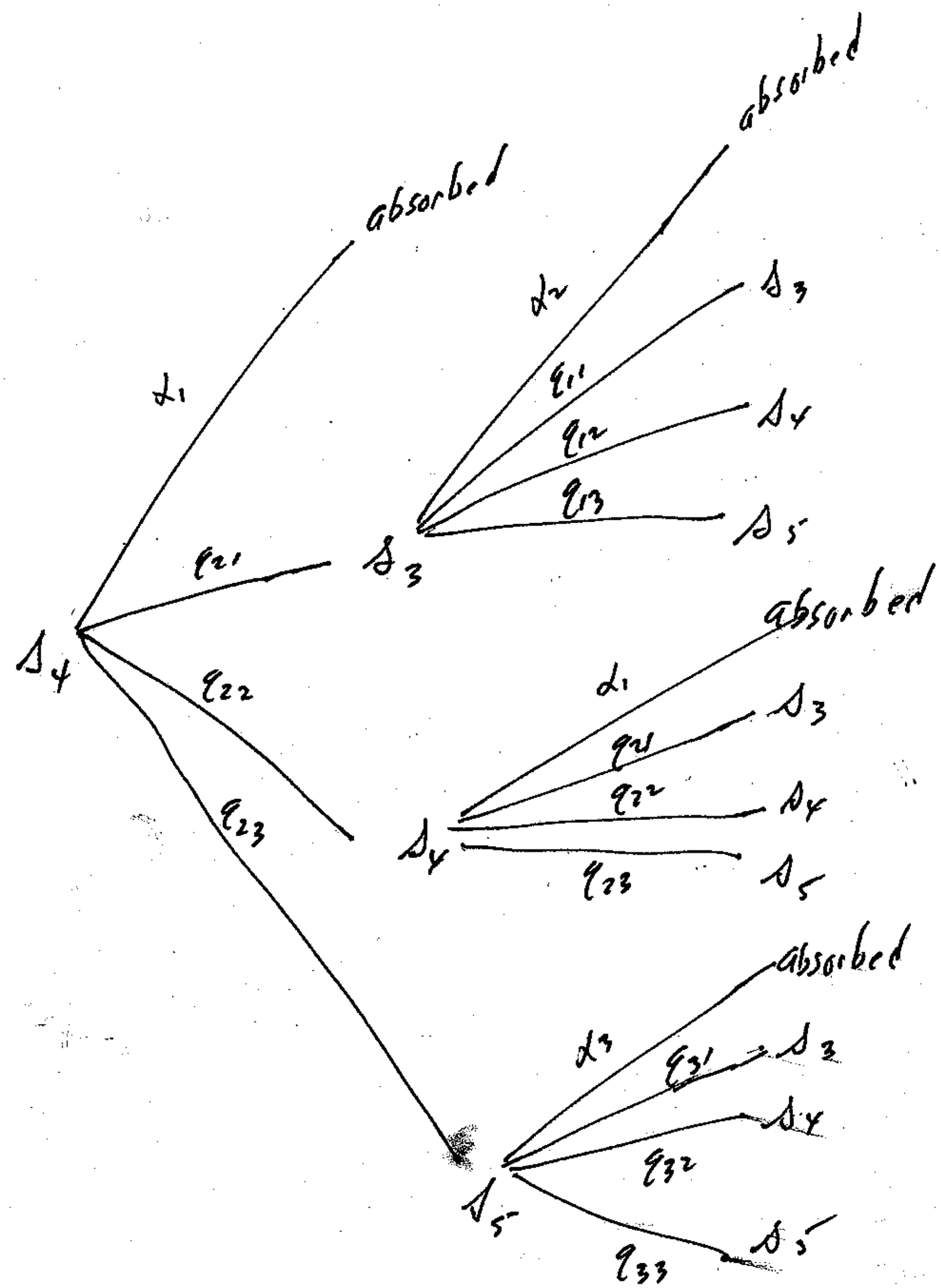
Suppose we begin in the i^{th} non-absorbing state and we seek the expected number of times we visit the j^{th} non-absorbing state in the 1st m trials. We will denote that expected number by $E_m(i,j)$.

We claim

$E_m(i,j)$ is the (i,j)-entry of $\mathbb{I} + \mathbb{Q} + \mathbb{Q}^2 + \cdots + \mathbb{Q}^m$

(2) That is, $E_m(i,j)$ is n_{ij} the (i,j)-entry of the fundamental matrix N.

As an example let's find $E_2(2,3)$.



$$E_1(2,3) = 1 \cdot q_{23}$$

$$E_2(2,3) = 1 \cdot q_{23} \cdot (1 - q_{23}) + 1 \cdot (q_{21} q_{13} + q_{22} \cdot q_{23}) + 2 \cdot q_{23} \cdot q_{33}$$

$$= q_{23} - q_{23}^2 + q_{21} q_{13} + q_{22} q_{23} + 2 q_{23} q_{33}$$

$$= q_{23} + [q_{21} q_{13} + q_{22} q_{23} + q_{23} q_{33}]$$

$$= q_{23} + q_{23}^{(2)} \leftarrow \text{(2,3)-entry of } I + Q + Q^2$$

Thm. n_{ij} in the fundamental matrix N is the expected number of times the process is in s_{k+j} given it began in s_{k+i} and ran until absorbed.

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Suppose process began in s_{k+i} and let $E_m(i,j)$ be the expected number of times it is in s_{k+j} in the first m trials. For now suppose $i \neq j$

$$E_1(i,j) = 1 \cdot q_{ij} = q_{ij}$$

$$E_2(i,j) = 1 \cdot q_{ij}(1 - q_{jj}) + 1 \cdot \sum_{i \neq j} q_{il} q_{lj} + 2 \cdot q_{ij} q_{jj}$$

$$= 1 \cdot q_{ij} - q_{ij} q_{jj} + 1 \sum_{i \neq j} q_{il} q_{lj} + 2 q_{ij} q_{jj}$$

$$= 1 \cdot q_{ij} + \sum q_{il} q_{lj} = q_{ij} + q_{ij}^{(2)}$$

← This is (i,j)-entry of $Q + Q^2$

By math induction we verify that

$$E_m(i,j) = q_{ij} + q_{ij}(2) + \cdots + q_{ij}(m), \quad i \neq j$$

If $i = j$ the system is in s_{k+j} initially so

$$E_m(i,j) = 1 + q_{ij} + \cdots + q_{ij}(m)$$

← (i,j)-entry of $I + Q + \cdots + Q^m$

So, $\lim_{m \to \infty} E_m(i,j) = n_{ij}$, the (i,j)-entry of N.