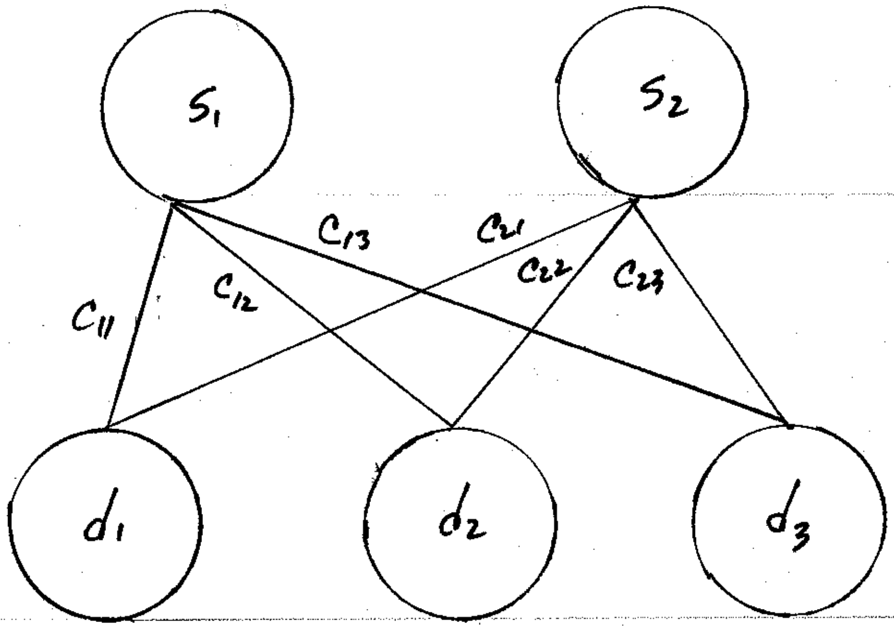
# Two-by-Three Transportation Problem




$d_i$ = demand at distribution center $i$
$s_i$ = supply at factory $i$
$c_{ij}$ = cost of moving one mobile home from factory $i$ to distribution center $j$.

~~#~~ $f_{ij}$ = the number of homes to be shipped from factory $i$ to distribution center $j$

Satisfy demand

$$f_{11} + f_{21} \geq d_1$$
$$f_{12} + f_{22} \geq d_2$$
$$f_{13} + f_{23} \geq d_3$$

Can't exceed supply

$$f_{11} + f_{12} + f_{13} \leq s_1$$
$$f_{21} + f_{22} + f_{23} \leq s_2$$

Calculate total moving cost

$$C(\bar{y}) = c_{11}f_{11} + c_{12}f_{12} + c_{13}f_{13} + c_{21}f_{21} + c_{22}f_{22} + c_{23}f_{23}$$

(we suppose $d_1 + d_2 + d_3 = s_1 + s_2$)

# A Mathematical Model

Undefined terms: data vector, allocation vector, cost vector

Axioms

$A_1$: A data vector is an ordered set of nonnegative integers $\bar{d} = (s_1, s_2, d_1, d_2, d_3)$.

$A_2$: $s_1 + s_2 = d_1 + d_2 + d_3$

$A_3$: An allocation vector is an ordered set of nonnegative integers $\bar{y} = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23})$.

$A_4$: A cost vector is an ordered set of six nonnegative rational numbers $\bar{c} = (c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23})$.

$A_5$: A data vector and an allocation vector satisfy the following inequalities:

$$\star \begin{cases} f_{11} + f_{21} = d_1, \quad f_{12} + f_{22} = d_2, \quad f_{13} + f_{23} = d_3 \\ f_{11} + f_{12} + f_{13} = s_1, \quad f_{21} + f_{22} + f_{23} = s_2 \end{cases}$$

Definitions The total transportation cost associated with the cost vector $\bar{c}$ and the allocation vector $\bar{y}$ is the number $\bar{y} \cdot \bar{c}$ defined by

$$\bar{y} \cdot \bar{c} = c_{11} f_{11} + c_{12} f_{12} + c_{13} f_{13} + c_{21} f_{21} + c_{22} f_{22} + c_{23} f_{23}$$

Problem of Interest: choose $\bar{y}$ to min $\bar{y} \cdot \bar{c}$ s.t. $(\star)$.

The problem:

Find an allocation vector $\bar{y} = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23})$ whose coordinates satisfy:

$$\begin{cases} f_{11} \phantom{+f_{12}+f_{13}} + f_{21} \phantom{+f_{22}+f_{23}} = d_1 \\ f_{12} + f_{22} = d_2 \\ f_{13} + f_{23} = d_3 \\ f_{11} + f_{12} + f_{13} = s_1 \\ f_{21} + f_{22} + f_{23} = s_2 \end{cases}$$

and such that

$$\bar{y} \cdot \bar{c} = (c_{11} f_{11} + c_{12} f_{12} + c_{13} f_{13} + c_{21} f_{21} + c_{22} f_{22} + c_{23} f_{23})$$

is minimized.

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & d_3 \\ 1 & 1 & 1 & 0 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 1 & 1 & 1 & s_2 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & -1 & -1 & -1 & s_1-d_1-d_2-d_3 \\ 0 & 0 & 0 & 1 & 1 & 1 & s_2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & -1 & d_1-s_2 \\ 0 & 1 & 0 & 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 & 1 & 1 & -s_1+d_1+d_2+d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & (s_1+s_2)-(d_1+d_2+d_3) \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & -1 & d_1-s_2 \\ 0 & 1 & 0 & 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 & 1 & 1 & s_2 \end{array}\right)$$

So, $\begin{cases} f_{11} = d_1 - s_2 + f_{22} + f_{23} \\ f_{12} = d_2 - f_{22} \\ f_{13} = d_3 - f_{23} \\ f_{21} = s_2 - f_{22} - f_{23} \end{cases}$

Because each $f_{ij} \geq 0$ $\quad (*) \begin{cases} 0 \leq f_{22} \leq d_2 \\ 0 \leq f_{23} \leq d_3 \\ s_2 - d_1 \leq f_{22} + f_{23} \leq s_2 \end{cases}$

So, we seek $f_{22}, f_{23}$ satisfying $(*)$ that minimizes $\vec{y} \cdot \vec{c}$.

We express $\bar{y}\cdot\bar{c}$ in terms of $f_{22}$ and $f_{23}$

$$\bar{y}\cdot\bar{c} = c_{11}(d_1 - s_2 + f_{22} + f_{23}) + c_{12}(d_2 - f_{22}) + c_{13}(d_3 - f_{23})$$
$$+ c_{21}(s_2 - f_{22} - f_{23}) + c_{22}f_{22} + c_{23}f_{23}.$$

$$= [c_{11}(d_1 - s_2) + c_{12}d_2 + c_{13}d_3 + c_{21}s_2]$$
$$+ [c_{11} - c_{12} - c_{21} + c_{22}]f_{22} + [c_{11} - c_{13} - c_{21} + c_{23}]f_{23}$$

$$= A + Bf_{22} + Cf_{23}$$

Where $A = [c_{11}(d_1 - s_1) + c_{12}d_2 + c_{13}d_3 + c_{21}s_2]$

$B = [c_{11} - c_{12} - c_{21} + c_{22}]$

$C = [c_{11} - c_{13} - c_{21} + c_{23}]$

Example let $d_1 = 6$, $d_2 = 9$, $d_3 = 10$, $s_1 = 15$, $s_2 = 10$

$c_{11} = 70$, $c_{12} = 70$, $c_{13} = 60$, $c_{21} = 80$, $c_{22} = 60$, $c_{23} = 80$.

Here $A = 1750$, $B = -20$, $C = 10$ and the constraints (*) become

$$(**)\begin{cases} 0 \le f_{22} \le 9 \\ 0 \le f_{23} \le 10 \\ 4 \le f_{22} + f_{23} \le 10 \end{cases}$$

So, we seek nonnegative $f_{22}$, $f_{23}$ satisfying (**) such that $\bar{y}\cdot\bar{c} = 1750 - 20f_{22} + 10f_{23}$ is minimum. Letting $f_{22} = 9$ and $f_{23} = 0$. $\bar{y}\cdot\bar{c}$ is minimized at 1570.

$\bar{\tau}_0 = (5, 0, 10, 1, 9, 0)$ is optimal allocation vector.