Some Kinds of Change and Some Models for Growth

Let $y = F(x)$ for $x \in [a, b]$.
Suppose, $y_1 = F(x_1)$ and $y_2 = F(x_2)$ for $x_1 < x_2$.

The *change in x*, $\Delta x = x_2 - x_1$, and
the *change in y*, $\Delta y = y_2 - y_1$.

The *average rate of change* in y with respect to x over the interval $(x_1, x_2]$ is the ratio

$$\Delta y/\Delta x.$$

The *percent change* in y over the interval $(x_1, x_2]$ is

$$100*\frac{[F(x_2) - F(x_1)]}{F(x_1)} = 100*\frac{[y_2 - y_1]}{y_1} = 100*\Delta y/y_1.$$

The *average proportionate growth rate* is the ratio

$$\frac{[\Delta y/\Delta x]}{F(x_1)} = \frac{[\Delta y/\Delta x]}{y_1}$$

The *average proportionate growth rate* is frequently called the *relative growth rate.*

Suppose we examine the size of a population at equally spaced times $t_0, t_1, t_2, \ldots$ and we denote the size of the population at time t_k by y_k, that is $y(t_k) = y_k$, $k = 0, 1, 2, \ldots$

<u>Example 1 (linear model):</u> **The size of a population increases by a fixed amount A in each time interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$ Here we have the relation**

$$y_{k+1} - y_k = A, \qquad k = 0, 1, 2, \ldots$$

or

$$y_{k+1} = y_k + A, \qquad k = 0, 1, 2, \ldots$$

In this example we can show that

$$y_k = Ak + x_0, \qquad k = 0, 1, 2, \ldots$$

<u>Example 2 (exponential model):</u> **The size of the population increases by a fixed multiple of the beginning population size in each time interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$ Here for some number r we have the relation**

$$y_{k+1} - y_k = ry_k, \qquad k = 0, 1, 2, \ldots$$

or

$$y_{k+1} = y_k + ry_k = (1 + r)\, y_k, \qquad k = 0, 1, 2, \ldots$$

In this example we can show that

$$y_k = (1 + r)^k\, y_0, \qquad k = 0, 1, 2, \ldots$$

Example 3 (logistic model): **Here we suppose that M is the maximum size of the population that can be maintained with the specific resources that are available and the relative growth rate is given by $c(M - y_k)$ for some constant c. Here we have the relation**

$$\frac{y_{k+1} - y_k}{y_k} = c(M - y_k), \qquad \mathbf{k = 0, 1, 2, \ldots}$$

or

$$y_{k+1} = y_k + c y_k (M - y_k), \qquad \mathbf{k = 0, 1, 2, \ldots}$$

Exercises

1. **Formulate linear and exponential models to fit the pattern of growth in the US population from 1790 to 1900.**

2. **Formulate a logistic model to fit the growth pattern for a sorghum plant.**

Year	Δyears	Years since 1790	US Population (millions)
1790		0	3.93
1800	10	10	5.31
1810	10	20	7.24
1820	10	30	9.64
1830	10	40	12.87
1840	10	50	17.07
1850	10	60	23.19
1860	10	70	31.44
1870	10	80	39.82
1880	10	90	50.16
1890	10	100	62.95
1900	10	110	76.00

Growth of a Sorghum Plant

Days After Planting t(k)	Weight in Grams w(k)
24	2
32	5
40	9
48	15
56	32
64	50
72	66
80	72
88	82
96	83