2.2 #1

We are given that for a fixed value of A

(2.1) $\quad x_{k+1} - x_k = A \quad$ for $k = 0, 1, 2, \cdots$

We hope to prove that it follows that

$$x_k = x_0 + kA \quad \text{for } k = 1, 2, \cdots$$

A proof by math induction follows.

(a) Basis Step

For $k=0$, $\quad x_1 - x_0 = A \quad$ applying (2.1)

$$x_1 = x_0 + A = x_0 + 1A$$

So, the proposition we hope to prove is true for $k=1$.

(b) Inductive Step

Suppose the proposition is true for some positive integer n. that is

$$x_n = x_0 + nA$$

So, applying (2.1) we obtain

$$x_{n+1} - x_n = A$$

$$\begin{aligned} x_{n+1} &= x_n + A \\ &= (x_0 + nA) + A \quad \text{by our inductive hypotheses} \\ &= x_0 + (n+1)A \end{aligned}$$

Hence if the proposition is true for $k=n$ it is true for $k=n+1$ also.

(a),(b) and PMI imply that

$$x_k = x_0 + kA \quad \text{for } k = 1, 2, \ldots$$

2.2 #2

Given $x_k = (1+r)^k x_0$ for $k = 1, 2, \dots$

Our time to double depends on r as follows:

$$2 = (1+r)^k$$

$$\ln 2 = k \ln(1+r)$$

$$k = \frac{\ln 2}{\ln(1+r)}$$

Let $n(r) = \dfrac{\ln 2}{\ln(1+r)}$ denote the time to double $0 \le 0.025 \le r \le 0.25$

Define $g(r) = \dfrac{72}{100r}$ $\quad 0.025 \le r \le 0.25$

We seek a range for r so that

$$\left|\frac{n(r) - g(r)}{n(r)}\right| \le 0.02.$$

We do some algebra (some steps omitted)

$$\left|\frac{n(r)-g(r)}{n(r)}\right| = \left|\frac{\frac{\ln 2}{\ln(1+r)} - \frac{72}{100r}}{\frac{\ln 2}{\ln(1+r)}}\right| = \left|1 - \frac{72}{100r} \cdot \frac{\ln(1+r)}{\ln 2}\right|$$

So we seek r so that

$$-0.02 \le 1 - \frac{72}{100r} \cdot \frac{\ln(1+r)}{\ln 2} \le 0.02$$

$$\vdots$$

$$\frac{98 \ln 2}{72} \le \frac{\ln(1+r)}{r} \le \frac{102 \ln 2}{72}$$

$$e^{\frac{98 \ln 2}{72}} \le (1+r)^{\frac{1}{r}} \le e^{\frac{102 \ln 2}{72}}$$

$$2.5688 \lessapprox (1+r)^{\frac{1}{r}} \lessapprox 2.6697$$

$$0.03695 \lessapprox r \lessapprox 0.12225 \Rightarrow$$ The relative error between $n(r)$ and $g(r)$ is less than 2% for r between 3.695% and 12.225%

7.7 #6

In this case we set $f(x) = bx(1-x^2)$, $b>0$ and our recurrence relation becomes
$x_{n+1} = f(x_n)$ where x_0 is given.

a) $\{x_n\}$ can be used as a population model if $0 \le x_n \le 1$ for all n. So, we must show a range of values for b that will guarantee that $0 \le x_n \le 1$. This means that we require that the maximum value of f must not exceed 1.

[Show that $f(x) \le 1$ provided that $b \le \frac{\sqrt{27}}{2} \approx 2.598$]

c) $\{x_n\} \xrightarrow[\text{monotonic}]{} x^* = \sqrt{\frac{b-1}{b}}$ for $1 < b \le 1.5$ [Determine the stable point x^*]

d) $\{x_n\} \to x^*$ not monotonic for $1.5 < b < 2$

Look at values of x_n to at least 4 decimal places