Ex 2.1 #2

Considering a single characteristic in a population that reproduces by selfing we will show that if the genotypic distribution is initially $[x \quad y \quad z]$, then in the n^{th} filial generation the distribution will be

$$\bar{d}_n = \left[x + y\cdot\frac{2^n-1}{2^{n+1}},\ y\cdot\frac{1}{2^n},\ y\cdot\frac{2^n-1}{2^{n+1}} + z\right]$$

Proof by PMI follows.

(*) Basis Step By Thm 2.2 the distribution in the 1^{st} generation will be

$$[x \quad y \quad z]\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} = \left[x + y\cdot\frac{1}{4},\ y\cdot\frac{1}{2},\ y\cdot\frac{1}{4} + z\right]$$

$$= \left[x + y\,\frac{2^1-1}{2^{1+1}},\ y\cdot\frac{1}{2^1},\ y\,\frac{2^1-1}{2^{1+1}} + z\right]$$

So, the proposition is true for $n=1$

(**) Induction Step

Suppose the proposition is true for $n=k$. That is, in the k^{th} generation the distribution is $\left[x + y\,\frac{2^k-1}{2^{k-1}},\ y\cdot\frac{1}{2^k},\ y\cdot\frac{2^k-1}{2^{k-1}} + z\right]$

Applying Thm 2.2 the distribution in the $(k+1)^{st}$ generation is

$$\left[x + y\,\frac{2^k-1}{2^{k-1}},\ y\,\frac{1}{2^k},\ y\,\frac{2^k-1}{2^{k-1}} + z\right]\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \left[\left(x + y\,\frac{2^k-1}{2^{k-1}}\right) + y\,\frac{1}{2^k}\cdot\frac{1}{4},\ y\,\frac{1}{2^k}\cdot\frac{1}{2},\ y\,\frac{1}{2^k}\cdot\frac{1}{4} + \left(y\,\frac{2^k-1}{2^{k-1}} + z\right)\right]$$

$$= \left[x + y\left(\frac{2^k-1}{2^{k-1}} + \frac{1}{2^{k+2}}\right),\ y\,\frac{1}{2^{k+1}},\ y\left(\frac{2^k-1}{2^{k-1}} + \frac{1}{2^{k+2}}\right) + z\right]$$

$$= \left[x + y\left(\frac{2^{k+2}-2+1}{2^{k+2}}\right),\ y\,\frac{1}{2^{k+1}},\ y\left(\frac{2^{k+2}-2+1}{2^{k+2}}\right) + z\right]$$

$$= \left[x + y\,\frac{2^{(k+1)+1}-1}{2^{(k+1)+1}},\ y\,\frac{1}{2^{k+1}},\ y\left(\frac{2^{(k+1)+1}-1}{2^{(k+1)+1}}\right) + z\right]$$

So, if the proposition is true for $n=k$, then it is true for $n=k+1$.

Hence, by (*), (**) and PMI the desired result follows.

Ex 2.1 / #4 Show that after the 1^{st} filial generation in a random mating population the genotypic frequencies associated with a single characteristic are constant.

Suppose $\bar{d}_0 = [x \quad y \quad z]$

Possible Pairings	$AA \times AA$,	$AA \times Aa$,	$AA \times aa$,	$Aa \times Aa$,	$Aa \times aa$,	$aa \times aa$
Freq	x^2	$2xy$	$2xz$	y^2	$2yz$	z^2
Distribution	$[1,0,0]$	$[\frac{1}{2},\frac{1}{2},0]$	$[0,1,0]$	$[\frac{1}{4},\frac{1}{2},\frac{1}{4}]$	$[0,\frac{1}{2},\frac{1}{2}]$	$[0,0,1]$

$$\bar{d}_1 = x^2[1,0,0] + 2xy[\tfrac{1}{2},\tfrac{1}{2},0] + 2xz[0,1,0] + y^2[\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{4}] + 2yz[0,\tfrac{1}{2},\tfrac{1}{2}] + z^2[0,0,1]$$
$$= [(x^2 + xy + \tfrac{1}{4}y^2), (xy + 2xz + \tfrac{1}{2}y^2 + yz), (\tfrac{1}{4}y^2 + yz + z^2)]$$
$$= [(x+\tfrac{1}{2}y)^2, 2(x+\tfrac{1}{2}y)(\tfrac{1}{2}y+z), (\tfrac{1}{2}y+z)^2]$$

Before calculating $\bar{d}_2$ we change variables as follows:

$$\alpha = (x + \tfrac{1}{2}y) \quad \beta = (z + \tfrac{1}{2}y)$$

So, $\bar{d}_1 = [\alpha^2, 2\alpha\beta, \beta^2]$ where $\alpha + \beta = 1$ (why?)

Applying the above procedure

$$\bar{d}_2 = [(\alpha^2+\alpha\beta)^2, 2(\alpha^2+\alpha\beta)(\alpha\beta+\beta^2), (\alpha\beta+\beta^2)^2]$$
$$= [\alpha^2(\alpha+\beta)^2, 2\alpha\beta(\alpha+\beta)^2, \beta^2(\alpha+\beta)^2]$$
$$= [\alpha^2, 2\alpha\beta, \beta^2]$$

Hence, $\bar{d}_2 = \bar{d}_1$. In like manner $\bar{d}_1 = \bar{d}_2 = \bar{d}_3 = \cdots$

Thus, after the 1^{st} filial generation the genotypic distributions (frequencies) associated with a single characteristic are constant. (With mating random.)