Some Function Concepts & Terminology – Corrected Version

A *function* f from a set X into a set Y, denoted by f: X → Y, associates each x ∈ X with a unique y ∈ Y. The unique y associated with x under f is denoted by f(x). The set X is called the *domain* of f, and the set Y is called the *codomain* of f. The set of all y ∈ Y such that y = f(x) for some x ∈ X is called the *range* of f.

A function f can be defined by a rule of correspondence.

Example 1: f: Q → Q might be defined by f(x) = 2x + 3 for each x ∈ Q.

A candidate for a function f: X → Y is *well-defined* iff for each $x_1, x_2 \in X$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

Example 2: f: R^{nonneg} → R defined by f(x) = y such that $y^2 = x$ is *not* well-defined because in this case f(4) could be either 2 or –2. Hence, f cannot be a function.

Example 3: **g**: Q → Q defined by $g(x) = 3(x + 1)^2$ is well-defined. To establish that g is well-defined we suppose $x_1, x_2 \in Q$ and $x_1 = x_2$. Now, $x_1 = x_2$ implies $x_1 + 1 = x_2 + 1$, which in turn implies $(x_1 + 1)^2 = (x_2 + 1)^2$, and that implies $3(x_1 + 1)^2 = 3(x_2 + 1)^2$. So, $g(x_1) = g(x_2)$, and **g** is well-defined.

A function f: X → Y is *onto* Y (or *surjective*) iff for each y ∈ Y there exists an x ∈ X such that f(x) = y.

Example 4: The function **g** of Example 3 is not onto Q because there exists no x ∈ Q such that **g**(x) = -1.

Example 5: The function f: Q → Q such the f(x) = 2x + 3 of Example 1 is onto Q. To prove that f is onto Q we suppose y ∈ Q. Now, f(x) = y if 2x + 3 = y. So, f(x) = y if x = ½(y – 3), and by properties of Q, ½(y – 3) ∈ Q. So, for this function, for any y ∈ Q there does exist an x = ½(y – 3) ∈ Q such that f(x) = y. Hence, f is onto Q.

A function f: X → Y is *one-to-one* (or *1-1* or *injective*) iff for each $x_1, x_2 \in X$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Equivalently, f: X → Y is *one-to-one* iff for each $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 6: The function **g** of Example 3 is not one-to-one because **g**(2) = 27 = **g**(-3).

Example 7: The function f Example 1 is one-to-one. To prove that f is one-to-one we suppose that $f(x_1) = f(x_2)$ for some arbitrary $x_1, x_2 \in Z$. It follows that $2x_1 + 3 = 2x_2 + 3$. So, $2x_1 = 2x_2$ and $x_1 = x_2$. So, for this function, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, and hence f is one-to one.

A function f: X → Y is *one-to-one and onto* Y (or *bijective*) iff f is both one-to-one and onto Y.

Example 8: The function f of Example 1 is one-to-one and onto Q.