MATH 210 Discrete Math – Session 20

We define a sequence $a_0, a_1, a_2, a_3, \ldots$ as follows:

(*) $a_0 = 3$
$a_k = a_{k-1} + 2k$ for all integers $k \geq 1$.

Write the first five terms of the sequence.

k	0	1	2	3	4
a_k					

Use mathematical induction to show that the terms of the sequence satisfy the formula
(**) $a_n = n^2 + n + 3$, for all integers $n \geq 0$.

(1) The formula (**) holds for n = 0: For n = 0 the formula gives $0^2 + 0 + 3 = 3$. But $a_0 = 3$ by the definition of the sequence. Hence the formula (**) holds for n = 0.

(2) If the formula (**) holds for n = k then it holds for n = k + 1: Let k be an integer with $k \geq 0$ and suppose that $a_k = k^2 + k + 3$. *[This is the inductive hypothesis.]* We must now show that $a_{k+1} = (k+1)^2 + (k+1) + 3$. But

$$\begin{aligned} a_{k+1} &= a_k + 2(k+1) && \text{by definition of the sequence} \\ &= (k^2 + k + 3) + 2(k+1) && \text{by the inductive hypothesis} \\ &= k^2 + 3k + 5 && \text{by regrouping} \\ &= (k^2 + 2k + 1) + (k+1) + 3 && \text{by regrouping} \\ &= (k+1)^2 + (k+1) + 3 && \text{by regrouping} \end{aligned}$$

[This is what was to be shown.] So, if the formula (**) holds for n = k, then the formula (**) holds for n = k + 1.

So, by the basis step (1) and the inductive step (2), and the PMI, the formula (**) holds for all terms of the sequence (*).

Prove: (***) $\forall n \in Z^+$ if $n \geq 5$, then $n^2 < 2^n$.

Basis Step:
The proposition (***) is true for n=5: $5^2 = 25 < 32 = 2^5$. *[This is the basis step.]*

Inductive Step:
If (***) is true for n = k then it is true for n = k+1: Suppose $k^2 \leq 2^k$ for an integer $k \geq 1$. We must show that it follows that $(k+1)^2 < 2^{k+1}$.Now,

$$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 && \text{by algebra} \\ &< 2^k + 2^k && \text{by inductive hypothesis } k^2 < 2^k \text{ and } (2k+1) < 2^k \text{ by Example 4.3.2.} \\ &< 2 \cdot 2^k = 2^{k+1} \end{aligned}$$

[This is what we needed to show.] By the basis and inductive steps and the PMI (***) is true.