7. Prove that for all sets A and B, $(A \cap B)^c = A^c \cup B^c$

Suppose A and B are sets.

(a) $(A \cap B)^c \subseteq A^c \cup B^c$: [We must show that $\forall x$ if $x \in (A \cap B)^c$ then $x \in A^c \cap B^c$.]

1) Suppose $x \in (A \cap B)^c$. [We must show that $x \in A^c \cup B^c$.]
By defintion of complement,

$x \notin A \cap B$.

But to say that $x \notin A \cap B$ means that
it is false that (x is in A and x is in B)
By DeMorgans law of logic, this implies
that x is not in A or x is not in B.
which can be written as $x \notin A$ or $x \notin B$

Hence $x \in A^c$ or $x \in B^c$ by definitionof complement
It follows by definition of union
that $x \in A^c \cup B^c$ [as was to be shown]. So
$(A \cap B)^c \subseteq A^c \cup B^c$ by definition of subset

(b) $A^c \cup B^c \subseteq (A \cap B)^c$: [We must show that $\forall x$, if $x \in A^c \cup B^c$ then $x \in (A \cap B)^c$].
Suppose $x \in A^c \cup B^c$. [We must show that $x \in (A \cap B)^c$]
By definition of intersection $x \in A^c$ or
$x \in B^c$, and by definition of complement,

$x \notin A$ or $x \notin B$.

In other words,

x is not in A or x is not in B.

By DeMorgans laws of logic this implies
it is false that (x is in A and x is in B),
Which can be written

$x \notin A \cap B$

by definition of intersection. Hence by
definition of complement $x \in (A \cap B)^c$ [as
was to be show] It follows that
$A^c \cup B^c \subseteq (A \cap B)^c$ by definition of subset.

Since both sets containments have
been proved, $(A \cap B)^c = A^c \cup B^c$ by definition
of set equality.