Proofs & Arguments – A Concise Summary

Theorems are often of the form

(*) If *p*, then *q*.

Or equivalently,

(*) $p \rightarrow q$

A *direct proof* assumes that *p* is true and then using *p* as well as axioms, definitions, and previously proved theorems, shows that *q* is true.

***Example:* The sum of two even numbers is even.**

***Proof:* Suppose m and n are arbitrary even integers. By the definition of even integer, there exists integers r and s such that m = 2r and n = 2s. Now, m + n = 2r + 2s = 2(r + s). But (r + s) is an integer; so m + n is even. Hence, if m and n are arbitrary even integers, then m + n is also an even integer. Consequently, the sum of two even numbers is even.**

A *proof by contradiction*, or *indirect proof*, establishes (*) by proving

(**) $(p \wedge \sim q) \rightarrow (r \wedge \sim r)$

***Example:* The sum of any rational number and any irrational number is irrational.**

***Proof*: Suppose that p is a [*fixed but arbitrary*] rational number and s is a [*fixed but arbitrary*] irrational number. Assume that r + s is rational. [*We will establish a contradiction.*] Since r is rational, -r is rational. Because the rationals are closed under addition (r + s) + -r is rational. But, (r + s) + -r = s; so s is rational. We now have s both irrational and rational – a contradiction! [*The assumption that r + s is rational lead to a contradiction.*] So, r + s must be irrational. We have shown that r rational and s irrational implies r + s is irrational. Consequently, the sum of any rational number and any irrational number is rational.**

A *proof by contra positive* establishes (*) by proving

(***) $\sim q \rightarrow \sim p$

***Example:* For any integer n, if n^2 is even, then n is even.**

***Proof:* Suppose n is an odd integer. [*That is n is not even.*] We have already shown that n^2 is odd. [*That is, n^2 is not even.*] Therefore, if n is odd, then n^2 is odd. Hence, [*employing the contra positive*] it follows that if n is even then, n^2 is even. Consequently, the square of any even integer is even.**