A *proof by cases* uses the rule of inference

(****) $[(r \vee s) \wedge (r \rightarrow q) \wedge (s \rightarrow q)] \rightarrow q$

Example: The square of any odd integer has the form 8m + 1 for some integer m.

Proof: Suppose n is an [*arbitrary but fixed*] odd integer. By the quotient-remainder theorem there exists a unique q such that exactly one of the following is true: $n = 4q$, $n = 4q + 1$, $n = 4q + 2$, or $n = 4q + 3$. [*By previously proved results*] $n = 4q + 1$ or $n = 4q + 3$ [*because 4q and 4q + 2 must both be even.*] [*We will consider both cases.*]
Case 1: $n = 4q + 1$
In this case $n^2 = 16q^2 + 8q + 1 = 8(2q^2 + q) + 1 = 8m + 1$ for some integer m.
Case 2: $n = 4q + 3$
In this case $n^2 = 16q^2 + 24q + 9 = 8(2q^2 + 3q + 1) + 1 = 8m + 1$ for some integer m.
So, in either case, $n^2 = 8m + 1$ for some integer m.
Therefore, if n is odd, n^2 has the form 8m + 1 for some integer m.
Consequently, the square of any odd integer has the form 8m + 1 for some integer m.

A *proof by mathematical induction* employs either the *Principle of Mathematical Induction* or the *Principle of Strong Mathematical Induction*.

Example: For any positive integer n, $1 + 2 + 3 + \ldots + n = \frac{1}{2}(n)(n + 1)$

Proof using the Principle of Mathematical Induction:
In this case we are trying to establish that $\forall n \in Z^+ P(n)$ where P(n) is given by
$P(n)$: $1 + 2 + 3 + \ldots + n = \frac{1}{2}(n)(n + 1)$.
Basis Step: P(1) is true because $\frac{1}{2}(1)(1 + 1) = 2/2 = 1$.
Inductive Step: Assume that P(k) is true for some $k \in Z^+$. [*This is our inductive hypothesis.*] [*We must now show that P(k + 1) is true.*] That is, we assume that

$1 + 2 + 3 + \ldots + k = \frac{1}{2}(k)(k + 1)$ for some [*arbitrary but fixed*] positive integer k.

Adding (k + 1) to both sides of the above equation, it follows that

$$1 + 2 + 3 + \ldots + k + (k + 1) = \tfrac{1}{2}(k)(k + 1) + (k + 1)$$

Applying algebra to the right hand side we obtain

$$\begin{aligned} 1 + 2 + 3 + \ldots + k + (k + 1) &= \tfrac{1}{2}(k)(k + 1) + (k + 1) \\ &= [\tfrac{1}{2}(k) + 1](k + 1) \\ &= \tfrac{1}{2}[k + 2](k + 1) \\ &= \tfrac{1}{2}(k + 1)[(k + 1) + 1] \end{aligned}$$

So, P(k + 1) or $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{1}{2}(k + 1)[(k + 1) + 1]$ is true.
We have shown that P(k) for some $k \in Z^+$ implies P(k + 1).
So, we have established that P(1) is true and that for any $k \in Z^+$ $P(k) \rightarrow P(k+1)$.
Consequently, by the Principle of Mathematical Induction, for any positive integer n, $1 + 2 + 3 + \ldots + n = \frac{1}{2}(n)(n + 1)$.