Some Kinds of Change and Some Models for Growth

Let $y = F(x)$ for $x \in [a, b]$.
Suppose, $y_1 = F(x_1)$ and $y_2 = F(x_2)$ for $x_1 < x_2$.

The *change in x*, $\Delta x = x_2 - x_1$, and
the *change in y*, $\Delta y = y_2 - y_1$.

The *average rate of change* in y with respect to x over the interval $(x_1, x_2]$ is the ratio

$$\Delta y/\Delta x.$$

The *percent change* in y over the interval $(x_1, x_2]$ is

$$100*\frac{[F(x_2) - F(x_1)]}{F(x_1)} = 100*\frac{[y_2 - y_1]}{y_1} = 100*\Delta y/y_1.$$

The *average proportionate growth rate* is the ratio

$$\frac{[\Delta y/\Delta x]}{F(x_1)} = \frac{[\Delta y/\Delta x]}{y_1}$$

The *average proportionate growth rate* is frequently called the *relative growth rate.*

Suppose we examine the size of a population at equally spaced times $t_0, t_1, t_2, \ldots$ and we denote the size of the population at time t_k by y_k, that is $y(t_k) = y_k$, $k = 0, 1, 2, \ldots$

<u>Example 1 (linear model):</u> **The size of a population increases by a fixed amount A in each time interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$ Here we have the relation**

$$y_{k+1} - y_k = A, \qquad k = 0, 1, 2, \ldots$$

or

$$y_{k+1} = y_k + A, \qquad k = 0, 1, 2, \ldots$$

In this example we can show that

$$y_k = Ak + x_0, \qquad k = 0, 1, 2, \ldots$$

<u>Example 2 (exponential model):</u> **The size of the population increases by a fixed multiple of the beginning population size in each time interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$ Here for some number r we have the relation**

$$y_{k+1} - y_k = ry_k, \qquad k = 0, 1, 2, \ldots$$

or

$$y_{k+1} = y_k + ry_k = (1 + r)\, y_k, \qquad k = 0, 1, 2, \ldots$$

In this example we can show that

$$y_k = (1 + r)^k\, y_0, \qquad k = 0, 1, 2, \ldots$$

Example 3 (logistic model): **Here we suppose that M is the maximum size of the population that can be maintained with the specific resources that are available and the relative growth rate is given by c(M – y_k) for some constant c. Here we have the relation**

$$\frac{y_{k+1} - y_k}{y_k} = c(M - y_k)\,, \qquad \mathbf{k = 0, 1, 2, \ldots}$$

or

$$y_{k+1} = y_k + c y_k (M - y_k)\,, \qquad \mathbf{k = 0, 1, 2, \ldots}$$

Exercises

1. **Formulate linear and exponential models to try to fit the pattern of growth in the US population from 1790 to 1900.**

2. **Formulate a logistic model to fit the growth pattern for a sorghum plant.**

US Census Data

n	Date	Years Since 1790 t(n)	Population (millions) P(n)
0	1790	0	3.929
1	1800	10	5.308
2	1810	20	7.240
3	1820	30	9.638
4	1830	40	12.866
5	1840	50	17.069
6	1850	60	23.192
7	1860	70	31.443
8	1870	80	38.558
9	1880	90	50.156
10	1890	100	62.948
11	1900	110	75.996

Growth of a Sorghum Plant

Days After Planting t(k)	Weight in Grams w(k)
24	2
32	5
40	9
48	15
56	32
64	50
72	66
80	72
88	82
96	83