Principle of Mathematical Induction

Let P(n) be a predicate that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. **P(a) is true.**
2. **For all integers k $\geq$ a, if P(k) is true then P(k + 1) is true.**

Then the statement

For all integers n $\geq$ a, P(n)

is true.

Example: **For any positive integer n, $1 + 3 + \ldots + (2n - 1) = n^2$**

Proof Using the Principle of Mathematical Induction:

In this case we are trying to establish that for each positive integer n, P(n) is true where P(n) is given by
P(n): $1 + 3 + \ldots + (2n-1) = n^2$.

Basis Step: **P(1) is true because $(2\cdot1 - 1) = 1 = 1^2$.**

Inductive Step: **Assume that P(k) is true for some positive integer k $\geq$ 1.**
[*This is our inductive hypothesis.*] [*We must now show that P(k + 1) is true.*]

That is, we assume that

$1 + 3 + \ldots + (2k - 1) = k^2$ for some [*arbitrary but fixed*] positive integer k.

Adding [2(k + 1) – 1] to both sides of the above equation, it follows that

$$1 + 3 + \ldots + (2k - 1) + [2(k + 1) - 1] = k^2 + [2(k + 1) - 1]$$

Applying algebra to the right hand side we obtain

$$\begin{aligned} 1 + 3 + \ldots + (2k - 1) + [2(k + 1) - 1] &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

So, P(k + 1), or $1 + 3 + \ldots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$, is true.
We have shown that P(k) for some integer k $\geq$ 1 implies P(k + 1).
We have now established that P(1) is true and that for any k $\geq$ 1, P(k) $\rightarrow$ P(k+1).

Consequently, by the Principle of Mathematical Induction,
For any positive integer n, $1 + 3 + \ldots + (2n - 1) = n^2$.