Some Notes on Reasoning, Logic, and Proof

We will consider the following topics:

- inductive reasoning
- representational reasoning
- mathematical statements
- deductive reasoning

Inductive reasoning is making a conclusion based on some specific observations. The conclusion drawn is called a ***generalization***.

Example: Given three consecutive integers, must at least one of them be a multiple of three?

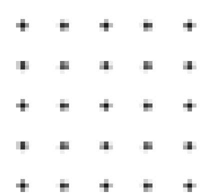
Given a relationship of some type, a ***representation*** of that relationship is something that captures the essential nature of the relationship by providing information sufficient for understanding and communicating the properties of the relationship.

Example: Consider the sums

$$1 = 1$$
$$1 + 2 + 1 = 4$$
$$1 + 2 + 3 + 2 + 1 = 9$$
$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$$

Using inductive reasoning what might we conclude about the n^{th} such sum?

How might the square array of dots represent the 5^{th} such sum?



Can we generalize?

A ***statement*** is a declarative sentence that is either true or false. Compound statements can be formed a variety of ways. Suppose p and q represent two statements.

- **And:** "p and q" is true only when both p and q are true, otherwise it is false.
- **Or:** "p or q" is false only when both p and q are false, otherwise it is true.
- **If … then:** "If p then q" is true when the truth of p guarantees the truth of q.
- **Not:** "not p" is true only when p is false, otherwise it is true.

"p and q" can be written as $p \wedge q$.
"p or q" can be written as $p \vee q$.
"If p then q" can be written as $p \rightarrow q$. This can also be read as "p implies q."
"not p" can be written as $\sim p$.

The rules above for determining the truth or falsity of a compound statement can be expressed in ***truth tables*** as follows. (T and F in the tables are called ***truth values***.)

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$\sim p$
T	T	T	T	T	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	F	T	T

Complete the truth table for $\sim p \vee q$.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	
T	F	F	
F	T	T	
F	F	T	

Exercises: If p, q, r, and s are the statements defined below, express each of the statements 1-5 below in words and give their respective truth values (T or F).

p: Annapolis is the capital of Maryland.
q: Annapolis is a city in Maryland.
r: Some pigs can fly.
s: $2 + 3 = 5$

1. $p \wedge r$
2. $p \vee s$
3. $r \rightarrow s$
4. $\sim r$
5. $\sim p \vee q$

If we have a set of statements that we know to be true, or we have accepted to be true, and on the basis of those statements we can argue that another statement must also be true, or accepted to be true, then we are using ***deductive reasoning***.

Those true statements we start with are called ***premises***, or ***hypotheses,*** of our argument, and the true statement we obtain is called the ***conclusion*** of the argument.

Example of Deductive Reasoning:

Premises:	If there is a tumor, then there is a need for surgery.
	There is a tumor.
Conclusion:	There is a need for surgery.

We can represent the argument symbolically if we let p represent the statement "There is a tumor" and q represent the statement "There is a need for surgery." Doing so we can state the ***rule of direct reasoning*** or the ***law of detachment*** as follows:

Rule of Direct Reasoning (Law of Detachment)

Premises: If p then q
p

Conclusion: $\therefore\ q$

Exercise: Complete the truth table for the rule of direct reasoning.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$

Example of Invalid Reasoning (Fallacy of the Converse)

Premises: If there is a tumor, then there is a need for surgery.
There is a need for surgery.
Conclusion: There is a tumor.

Exercise: Complete the truth table this example of invalid reasoning.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$[(p \rightarrow q) \wedge q] \rightarrow p$

Example of Indirect Deductive Reasoning

Premises: If there is a tumor, then there is a need for surgery.
There is not a need for surgery.
Conclusion: There is not a tumor.

Rule of Indirect Reasoning (Law of Contraposition)

Premises: If p then q
$\sim q$

Conclusion: $\therefore\ \sim p$

Exercise: Complete the truth table this example of invalid reasoning.

p	q	$p \rightarrow q$	$\sim q$	$(p \rightarrow q) \wedge \sim q$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

Example of Transitive Deductive Reasoning

Premises: If there is a tumor, then there is a need for surgery.
If there is a need for surgery, then Dr. Smith will operate.
Conclusion: If there is a tumor, then Dr. Smith will operate.

We can represent the argument symbolically if we let *p* represent the statement "There is a tumor," *q* represent the statement "There is a need for surgery" and *r* represent the statement Dr. Smith will operate." Doing so we can state the ***rule of transitive reasoning*** as follows:

Rule of Transitive Reasoning (Law of Syllogism)

Premises: If *p* then *q*
If *q* then *r*

Conclusion: ∴ If *p* then *r*

Exercise: Complete the truth table the rule of transitive reasoning.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r]) \to (p \to r)$

Exercises: Translate the following arguments into symbolic form and determine whether or not each argument is valid.

a. **Premises:** If there is a tumor, then there is a need for surgery.
If there is a need for surgery, then Dr. Smith will operate.
There is a tumor.
Conclusion: ∴ Dr. Smith will operate.

b. **Premises:** If there is a tumor, then there is a need for surgery.
If there is a need for surgery, then Dr. Smith will operate.
There is a not a tumor.
Conclusion: ∴ Dr. Smith will not operate.

c. **Premises:** If there is a tumor, then there is a need for surgery.
If there is a need for surgery, then Dr. Smith will operate.
Dr. Smith will operate.
Conclusion: ∴ There is a tumor.

d. **Premises:** If there is a tumor, then there is a need for surgery.
If there is a need for surgery, then Dr. Smith will operate.
Dr. Smith will not operate.
Conclusion: ∴ There is not a tumor.

Some Standard Forms for Arguments

Valid Arguments

	Law of Detachment	*Law of Contraposition*	*Law of Syllogism*
	$p \rightarrow q$	$p \rightarrow q$	$p \rightarrow q$
	$\underline{p}$	$\underline{\sim q}$	$\underline{q \rightarrow r}$
	$\therefore q$	$\therefore \sim p$	$\therefore p \rightarrow r$

	Disjunctive Syllogism	*Reductio Absurdum*
	$p \vee q$	$p \wedge \sim q$
	$\underline{\sim p}$	$\underline{[p \wedge \sim q] \rightarrow (r \wedge \sim r]}$
	q	q

Invalid Arguments

	Fallacy of the Converse	*Fallacy of the Inverse*
	$p \rightarrow q$	$p \rightarrow q$
	$\underline{q}$	$\underline{\sim p}$
	p	$\sim q$

The types of arguments considered up to now are known as ***symbolic arguments.*** Their validity or nonvalidity was determined using truth tables or comparison to standard forms. Another type of argument, sometimes called a ***syllogistic argument***, may have its validity tested using set, or Euler, diagrams. In syllogistic logic the deductive process considers the relationship between the following four types of statements.

All __________ are _________.
No __________ are__________.
Some_________are__________.
Some________are not_______.

The words *all*, *none* (or *no*), and *some* are called ***quantifiers***. Consider the following quantified statements and their negations.

All humans have red hair.	*Some human(s) does(do) not have red hair.*
No human has red hair.	*Some human has red hair.*
Some human has red hair.	*No human has red hair.*
Some human(s) does(do) not have red hair.	*All humans have red hair.*

Exercises: Write negations of each sentence.

a. Some birds can swim.

b. All men are tall.

c. No even number is prime.

d. Some birds cannot swim.

We can illustrate quantified sentences using set, or Euler, diagrams.

All P's are Q's. No P's are Q's. Some P's are Q's. Some P's are not Q's.

Exercises: In each case use a set, or Euler, diagram to determine whether the argument is valid or is a fallacy.

a. All frogs are green.
Prince is a frog.
$\therefore$ Prince is green.

b. All frogs are green.
Prince is green.
$\therefore$ Prince is a frog.

c. Some rectangles are squares.
All squares are parallelograms.
$\therefore$ Some rectangles are parallelograms.

d. All pigs are red.
Everything red can fly.
$\therefore$ All pigs can fly.

Examples of Deductive Reasoning in Mathematics

Claim: If a natural number n is a multiple of 3, then n^2 is a multiple of 9.

Proof: Suppose n is a natural number that is a multiple of 3. It follows that for some natural number k, n = 3k. So, $n^2 = (3k)^2 = 9k^2$. Hence, n^2 is a multiple of 9. Consequently, if a natural number n is a multiple of 3, then n^2 is a multiple of 9.

Claim: If n is a natural number and n^2 is odd, then n is also odd.

Proof: Suppose n is a natural number and n^2 is odd. Now suppose also that n is even. Since n is even n = 2k for some natural number k. So we have $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Hence it follows that n^2 is even. We have a contradiction. So, n cannot be even. Hence n is odd. Consequently, if n is a natural number and n^2 is odd, then n is also odd.

Exercises: Use deductive reasoning to establish the following claims.

a. If the sum of the digits of a three-digit number is a multiple of 9, then the number is a multiple of 9 also.

b. There is no largest prime number.

c. $\sqrt{2}$ is an irrational number.

Challenge Exercise: Can we make and support a claim relative to whether or not each square of an odd number is one more than a multiple of 8?