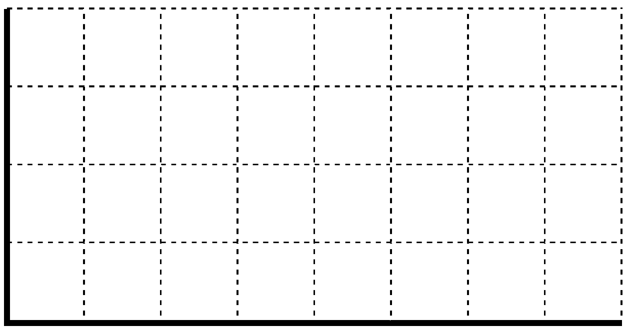
# Affine Transformations & Change of Coordinates

An ***affine transformation*** $T$: $R^n \rightarrow R^n$ has the form $T(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ where A is an $n$ $x$ $n$ matrix and $\mathbf{b} \in R^n$.

*Example 1.* Suppose for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Suppose S is the square with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Sketch both S and the image of S, $T$(S), under the transformation.

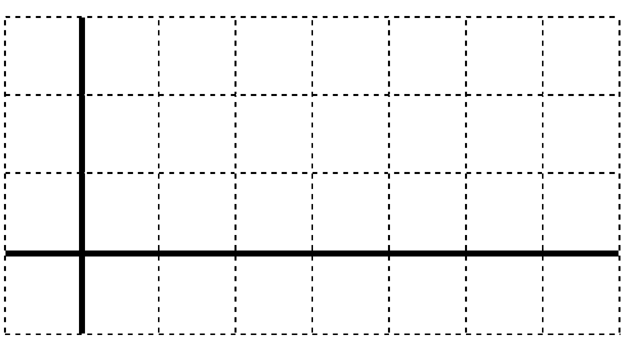


Explain why, in general, affine transformations are not linear transformations.

An ***isometry*** preserves distances. So, a transformation $T$: $R^n \rightarrow R^n$ will be an isometry provided $\|\mathbf{u} - \mathbf{v}\| = \| T(\mathbf{u}) - T(\mathbf{v}) \|$ for all $\mathbf{u}, \mathbf{v} \in R^n$.

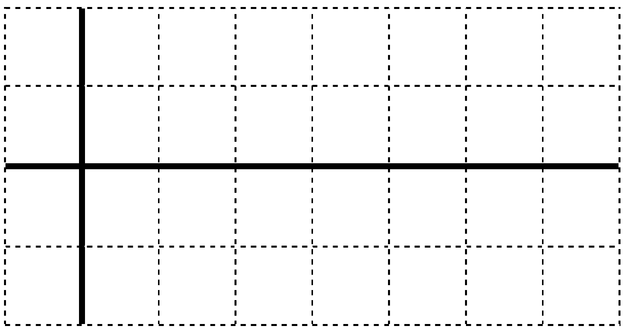
Is the transformation of Example 1 an isometry?

*Example 2.* Suppose for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose S is the square with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Sketch both S and the image of S, $T$(S), under the transformation.



Is the transformation of Example 2 an isometry?

*Example 3.* Let S be the triangle with vertices $\mathbf{A} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find an affine transformation $T$ such that $T(\mathbf{A}) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \mathbf{A'}$, $T(\mathbf{B}) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \mathbf{B'}$, and $T(\mathbf{C}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \mathbf{C'}$.
Is your $T$ an isometry? As geometric figures, how are S and $T$(S) related?

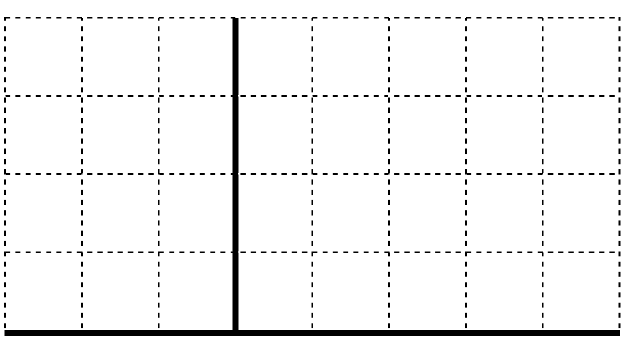


*Example 4.* Consider the line in $E^2$ that has equation y = 2x relative to the standard coordinate system – that is relative to the standard basis for $R^2$ which is $S = \{\mathbf{e_1}, \mathbf{e_2}\}$. Suppose the basis vectors for a new coordinate system $B$ are $\mathbf{b_1} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{b_2} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.
What will be the equation of the line relative to the $B$-coordinate system?

Let B = $[\mathbf{b_1}\ \ \mathbf{b_2}]$. For any point with standard coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$ we denote its $B$-coordinates by $\begin{bmatrix} x' \\ y' \end{bmatrix}$.
We know that $B\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. So, $x = \frac{1}{\sqrt{5}}x' + \frac{-2}{\sqrt{5}}y'$ and $y = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'$.

Substituting for x and y in the original equation we get

$(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y') = 2(\frac{1}{\sqrt{5}}x' + \frac{-2}{\sqrt{5}}y')$. Solving for $y'$ we obtain the equation for the line in the B-coordinate system.



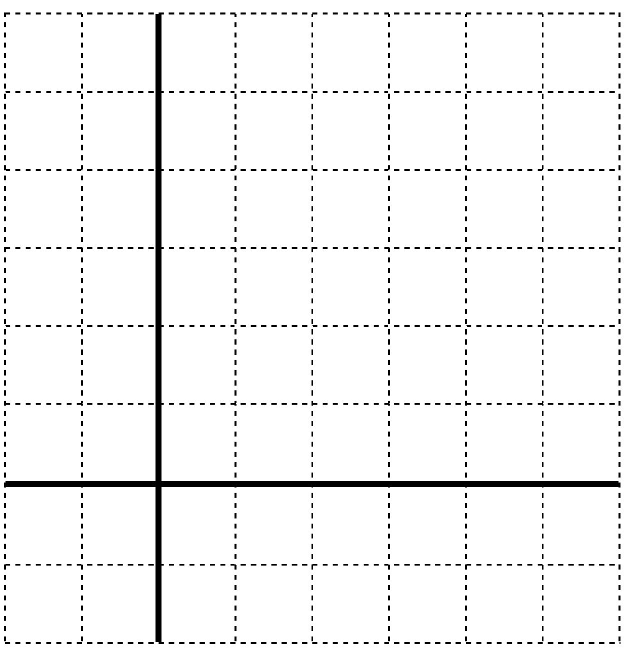
*Example 5.* The basis vectors for a coordinate system *C* are $\mathbf{c_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{c_2} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Consider the curve in $E^2$ that has equation y' = $(x')^2 + 1$ relative to the C-coordinate system. What will be the equation of the curve relative to the standard coordinate system?

Let C = [$\mathbf{c_1}$ $\mathbf{c_2}$]. For any point with standard coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$ we denote its *C*-coordinates by $\begin{bmatrix} x' \\ y' \end{bmatrix}$.

We know that $C\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Therefore $C^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$. Where $C^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

So, $x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$ and $y' = \frac{-1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$. Substituting for *x'* and *y'* in the original equation we get $(\frac{-1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y) = (\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y)^2 + 1$. Expressing the equation in general quadratic form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ we obtain the following:



We note the discriminant test from a calculus text that tells us that, except for a few degenerate cases an equation in the general quadratic form represents:

a. a parabola if $b^2 - 4ac = 0$,
b. an ellipse if $b^2 - 4ac < 0$, and
c. a hyperbola if $b^2 - 4ac > 0$.

Given an equation of a curve in the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ relative to the standard coordinate system, it can be shown that the cross product (xy) term can be eliminated by transforming coordinates by rotating the coordinate axes through a counterclockwise rotation of α where $\tan 2\alpha = b/(a - c)$. Equivalently, we write the equation in terms of new coordinates with respect to the basis $B = \{\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}\}$.