Linear Algebra

Section 6.1 Revisited

For $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ the <u>inner product</u> of $\bar{x}$ and $\bar{y}$, denoted by $\bar{x} \cdot \bar{y}$ is the scalar defined by $\bar{x} \cdot \bar{y} = \sum_{i=1}^{n} x_i y_i$. This product is also called the <u>dot product</u> or <u>scalar product</u> of $\bar{x}$ and $\bar{y}$.

<u>Example 1</u> Let $\bar{x} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, $\bar{y} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$ and calculate $\bar{x} \cdot \bar{y}$.

For $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ the <u>length</u> (or <u>norm</u>) of $\bar{x}$, denoted by $\|x\|$, is the scalar defined by $\|\bar{x}\| = \sqrt{\bar{x} \cdot \bar{x}}$

<u>Example 2</u> For $\bar{x} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, find $\|\bar{x}\|$.

A vector of length 1 is called a <u>unit vector</u>.

<u>Example 3.</u> Find a unit vector in the subspace spanned by $\begin{bmatrix} 5 \\ 12 \end{bmatrix}$.

For $\bar{x}, \bar{y} \in \mathbb{R}^n$, the distance between $\bar{x}$ and $\bar{y}$, denoted by $\text{dist}(\bar{x}, \bar{y})$ is defined by
$\text{dist}(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|$.

<u>Example 4</u> If $\bar{x} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$, find $\text{dist}(\bar{x}, \bar{y})$.

If $\bar{x}$ and $\bar{y}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have seen that the angle θ between $\bar{x}$ and $\bar{y}$ is related to the inner product by the formula

$$\bar{x} \cdot \bar{y} = \|\bar{x}\| \|\bar{y}\| \cos\theta.$$

<u>Example 5</u> What can we say about the angle between the vectors $\bar{x} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$?

Two vectors $\bar{x}$ and $\bar{y} \in \mathbb{R}^n$ are <u>orthoganal</u> if $\bar{x} \cdot \bar{y} = 0$.

<u>Example 6</u> Let $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$.
(a) Specify a basis for Col A.
(b) Specify a basis for Nul A^T.
(c) Show that any nonzero vector in Col A is orthogonal to any nonzero vector in Nul A^T.

Orthogonal Sets

A set of vectors $\{\bar{u}_1, \ldots, \bar{u}_p\} \in \mathbb{R}^n$ is called an <u>orthogonal</u> <u>set</u> if each pair of distinct vectors in the set is orthogonal. That is, if $\bar{u}_i \cdot \bar{u}_j = 0$ whenever $i \neq j$.

An <u>orthogonal</u> <u>basis</u> for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

<u>Example 7</u> Show that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$, and then express $\bar{x} = \begin{bmatrix} 8 \\ -4 \\ 3 \end{bmatrix}$ as a linear combination of those basis vectors.

A set $\{\bar{u}_1, \ldots, \bar{u}_p\}$ is an <u>orthonomal</u> <u>set</u> if it is an orthogonal set of unit vectors. If such a set is also a basis for a subspace, it is called an <u>orthonormal</u> <u>basis</u> for the subspace.

<u>Example 8</u> Show that $\left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.