Owls & Rats
(Section 5.6)

We denote the owl and rat populations at time k (months) by

$$\bar{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$$

where O_k is the number of owls and R_k the number of rats in the 1000's.

We suppose
$$\begin{cases} O_{k+1} = 0.5\,O_k + 0.4\,R_k \\ R_{k+1} = -0.104\,O_k + 1.1\,R_k \end{cases}$$

That is, $\bar{x}_{k+1} = A\,\bar{x}_k$ where $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$.

Note, if $\bar{x}_0 = \begin{bmatrix} O_0 \\ R_0 \end{bmatrix}$ is our initial population vector,

$$\begin{aligned}
\bar{x}_1 &= A\bar{x}_0 \\
\bar{x}_2 &= A\bar{x}_1 = A(A\bar{x}_1) = A^2 x_0 \\
\bar{x}_3 &= A\bar{x}_2 = A(A^2\bar{x}_0) = A^3\bar{x}_0 \\
&\vdots \\
\bar{x}_k &= A^k\bar{x}_0 \\
&\vdots
\end{aligned}$$

A has eigenvalues $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$ with corresponding eigenvectors $\begin{bmatrix} 10 \\ 13 \end{bmatrix} = \bar{v}_1$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \bar{v}_2$.

Any initial population vector $\bar{x}_0$ can be written as

$$\bar{x}_0 = c_1 \bar{v}_1 + c_2 \bar{v}_2 \quad \text{for some } c_1, c_2 \in \mathbb{R}.$$

Then,

$$\bar{x}_1 = A\bar{x}_0 = A(c_1\bar{v}_1 + c_2\bar{v}_2) = c_1(1.02)^1\bar{v}_1 + c_2(.58)\bar{v}_2$$

$$\bar{x}_2 = A\bar{x}_1 = c_1(1.02)^2\bar{v}_1 + c_2(.58)^2\bar{v}_2$$

$$\vdots$$

$$\bar{x}_k = c_1(1.02)^k\bar{v}_1 + c_2(.58)^k\bar{v}_2$$

$$= c_1(1.02)^k\begin{bmatrix}10\\13\end{bmatrix} + c_2(.58)^k\begin{bmatrix}5\\1\end{bmatrix}$$

We can see that for large values of k

$$\bar{x}_k \approx c_1(1.02)^k\begin{bmatrix}10\\13\end{bmatrix}$$

and

$$\bar{x}_{k+1} \approx c_1(1.02)^{k+1}\begin{bmatrix}10\\13\end{bmatrix} = (1.02)\left[c_1(1.02)^k\begin{bmatrix}10\\13\end{bmatrix}\right] \approx 1.02\,\bar{x}_k$$

That is,

$$\bar{x}_{k+1} \approx 1.02\,\bar{x}_k$$

So, in the long run, ...