Quadratic Forms

A quadratic form on $\mathbb{R}^n$ is a function Q whose value at $\bar{x} \in \mathbb{R}^n$ can be computed by an expression of the form

$$Q(\bar{x}) = \bar{x}^T A \bar{x} \quad \text{where } A \text{ is}$$

an $n \times n$ symmetric matrix. A is called the matrix of the quadratic form.

Example $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. We compute $\bar{x}^T A \bar{x}$ for $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\bar{x}^T A \bar{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2$$

Example $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. We now compute $\bar{x}^T A \bar{x}$ for $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

$$Q(\bar{x}) = \bar{x}^T A \bar{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$Q(\bar{x}) = 3x_1^2 - 4x_1y_1 + 7x_2^2$$

$$Q\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right) = 27 + 12 + 7 = 46$$

Change of Variable

For $\bar{x} \in \mathbb{R}^n$ a <u>change of variable</u> is an equation of the form

(*) $\quad \bar{x} = P\bar{y} \quad$ or $\quad \bar{y} = P^{-1}\bar{x}$

where P is an invertible matrix and $\bar{y} \in \mathbb{R}^n$.

Note that $\bar{y}$ is the coordinate vector of $\bar{x}$ relative to the basis of $\mathbb{R}^n$ determined by the columns of P.

If the change of variable (1) is made in a quadratic form $\bar{x}^T A \bar{x}$ we have

$$\bar{x}^T A \bar{x} = (P\bar{y})^T A (P\bar{y}) = y^T P^T A P \bar{y} = y^T (P^T A P)\bar{y}$$

and the new matrix of the quadratic form is $P^T A P$.

If P orthogonally diagonalizes A, then $P^T = P^{-1}$ and $P^T A P = P^{-1} A P = D$, and the matrix of the new quadratic form is diagonal.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

<u>Example</u> $Q(\bar{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$

$$= \bar{x}^T \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} \bar{x}$$

We orthogonally diagonalize A.
Its eigenvalues are $\lambda = 3$ and $\lambda = -7$.

The associated unit eigenvalues are

$\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ respectively.

Let $P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$

$$\Rightarrow A = PDP^{-1} \Rightarrow D = P^{-1}AP = P^TAP$$

A change of variable is

$$\bar{x} = P\bar{y}$$

$$\begin{aligned}
\Rightarrow x_1^2 - 8x_1y_1 - 5x_2^2 &= \bar{x}^TA\bar{x} = (P\bar{y})^TA(P\bar{y}) \\
&= y^TP^TAP\bar{y} \\
&= \bar{y}^TD\bar{y} \\
&= [y_1 \ y_2]\begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= [y_1 \ y_2]\begin{bmatrix} 3y_1 \\ -7y_2 \end{bmatrix} \\
&= 3y_1^2 - 7y_2^2
\end{aligned}$$

Note, if $\bar{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $Q\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) = 16$

$\bar{y} = P\bar{x} = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \Rightarrow \bar{y}^TD\bar{y} = 16$ also.