Determinates Revisited

Suppose A is an $n \times n$ matrix

A_{ij} is the submatrix of A formed by deleting the i^{th} row and j^{th} column of A

Definition For $n \geq 2$ the determinate of A is defined as follows:

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Suppose $A = [a_{ij}]_{n \times n}$, the (i,j)-cofactor of A is the number C_{ij} given by $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Then

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

Theorem 1 (Section 3.1)

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any } 1 \leq i \leq n$$

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Theorem 10 (Section 3.3)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by $A_{2\times 2}$. If S is a parallelogram in $\mathbb{R}^2$ then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation determined by $A_{3\times 3}$, then if S is a parallelepiped in $\mathbb{R}^3$ then

$$\{\text{Volume of } T(S)\} = |\det A| \cdot \{\text{Volume of } S\}$$

Remark Thm 10 holds whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

Suppose A is $n \times n$ and $\bar{b} \in \mathbb{R}^n$

Let $A_i(\bar{b})$ be the matrix obtained from A by replacing column i by the vector $\bar{b}$.

$$A_i(\bar{b}) = [\bar{a}_1 \ \cdots \ \underset{\text{col. } i}{\bar{b}} \ \cdots \ \bar{a}_n]$$

Theorem 7 Cramer's Rule

Let $A_{n\times n}$ be invertible and $\bar{b} \in \mathbb{R}^n$ the unique solution $\bar{x}$ of $A\bar{x} = \bar{b}$ has entries given by

$$x_i = \frac{\det A_i(\bar{b})}{\det A}, \quad i = 1, 2, \dots, n$$

Theorem 8: If $A_{n\times n}$ is invertible, then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

where $\operatorname{adj} A$ is called adjugate of A and is defined by

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where the C_{ij}'s are cofactors.