Theorem 6 (Section 6.2) An $m \times n$ matrix U has orthonormal columns iff $U^TU = I_n$.

Proof (Case where U is 3 x 3)

Let $U = [\,\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \mathbf{u}_3\,]$

(1) $U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] =$

The columns of U are orthogonal iff

(2)

The columns of U all have unit length iff

(3)

The desired result follows from (1), (2), and (3).

Diagonalization of Symmetric Matrices A ***symmetric matrix*** is a matrix A such that $A^T = A$.

If possible diagonalize the matrix A = $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. (A's eigenvalues are 4 and 2.)

Theorem 1 (Section 7.1) If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Proof Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be eigenvectors that correspond to distinct eigenvalues λ_1 and λ_2. We hope to show $\mathbf{v_1} \cdot \mathbf{v_2} = 0$.

$$\lambda_1 \mathbf{v_1} \cdot \mathbf{v_2} = (\lambda_1 \mathbf{v_1})^T \mathbf{v_2} = (A\mathbf{v_1})^T \mathbf{v_2} = (\mathbf{v_1}^T A^T)\mathbf{v_2} = \mathbf{v_1}^T (A\mathbf{v_2}) = \mathbf{v_1}^T (\lambda_2 \mathbf{v_2}) = \lambda_2 \mathbf{v_1}^T \mathbf{v_2}$$

So, $\lambda_1 \mathbf{v_1} \cdot \mathbf{v_2} - \lambda_2 \mathbf{v_1}^T \mathbf{v_2} = (\lambda_1 - \lambda_2)\mathbf{v_1} \cdot \mathbf{v_2} = 0$.
But $\lambda_1 - \lambda_2 \neq 0$, so $\mathbf{v_1} \cdot \mathbf{v_2} = 0$.

A matrix A is said to be orthogonally diagonizable is there exists an orthogonal matrix P with $P^{-1} = P^T$ and a diagonal matrix D such that $A = PDP^T = PDP^{-1}$.

Note that if A is orthogonally diagonalizable then

$$A^T = (PDP^T)^T = P^{TT} D^T P^T = PDP^T = A$$

Thus A is symmetric.

The Spectral Theorem for Symmetric Matrices (Section 7.1)

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.
b. The dimension of each eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
c. The eigenspaces are mutually orthogonal.
d. A is orthogonally diagonalizable.