Elementary Matrices

An *elementary matrix* is one that is obtained by performing a single elementary row operation on an identity matrix.

Examples

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} = E_1$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\frac{1}{2}R_2 \to R_2}{\sim} \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} = E_2$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{-3R_2 + R_3 \to R_3}{\sim} \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} = E_3$$

Now calculate the following matrix products where

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 3 & 7 \end{pmatrix}.$$

$$E_1 A = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} = B$$

$$E_2 B = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} = C$$

$$E_3 C = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

$$E_3\left(E_2\left(E_1 A\right)\right) = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

① Is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 7 \end{bmatrix}$ a non-singular matrix? If so, determine A^{-1}.

② Find elementary matrices E_1, E_2, E_3 such that $E_3 E_2 E_1 A = I_3$, if such matrices exist. If no such matrices exist, explain why.

If E_1, E_2, E_3 exist, calculate $E_3 E_2 E_1 I_3$.

③ If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 7 \end{bmatrix}$ is T onto $\mathbb{R}^3$? Is T 1-1? If T is 1-1, specify a rule for T^{-1}.