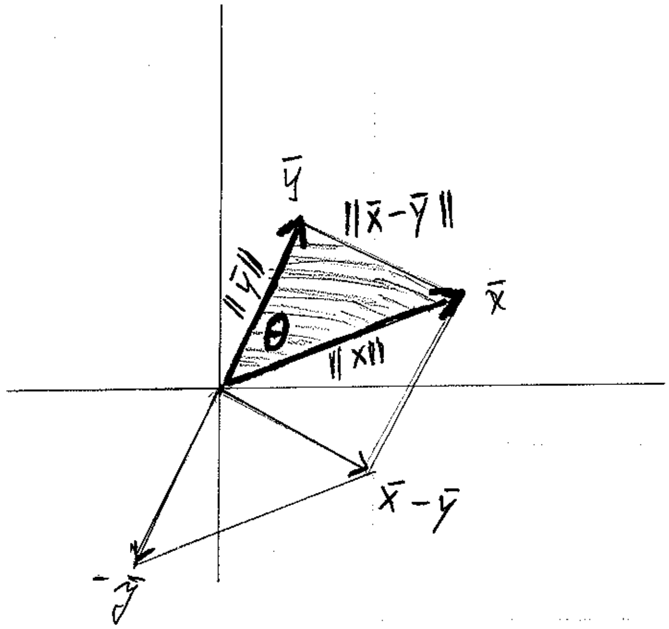
Angle between Vectors & Inner Product (Session #34 11/15/06)



$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

length of $\bar{x}$, denoted by $\|\bar{x}\|$, is $\sqrt{x_1^2 + x_2^2}$

length of $\bar{y}$, denoted by $\|\bar{y}\|$, is $\sqrt{y_1^2 + y_2^2}$

length of $\bar{x}-\bar{y}$, denoted by $\|\bar{x}-\bar{y}\|$, is $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

By the <u>law of cosines</u>

$$\|\bar{x}-\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2\|\bar{x}\|\|\bar{y}\|\cos\theta$$

Applying some algebra,

$$(x_1-y_1)^2 + (x_2-y_2)^2 = (x_1^2+x_2^2) + (y_1+y_2)^2 - 2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}\cos\theta$$

$$-2x_1y_1 - 2x_2y_2 = -2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}\cos\theta$$

So, $$\boxed{\cos\theta = \frac{x_1y_1 + x_2y_2}{\|\bar{x}\|\|\bar{y}\|}} \qquad (1)$$

For vectors $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$, the <u>dot product</u> or <u>inner product</u> or <u>scalar product</u> of $\bar{x}$ and $\bar{y}$ denoted by $\bar{x}\cdot\bar{y}$ is defined by $\bar{x}\cdot\bar{y} = \sum_{i=1}^{n} x_i y_i$.

Hence, $$\boxed{\cos\theta = \frac{\bar{x}\cdot\bar{y}}{\|\bar{x}\|\|\bar{y}\|}} \qquad (2)$$

Of course it also follows that for $\bar{x}, \bar{y} \in \mathbb{R}^n$,

$$\|\bar{x}\| = (\bar{x} \cdot \bar{x})^{1/2} \quad \text{and} \quad \|\bar{y}\| = (\bar{y} \cdot \bar{y})^{1/2}$$

<u>Example 1</u> For $\bar{x} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ find $\|\bar{x}\|$, $\|\bar{y}\|$, and $\cos\theta$ where θ is the angle formed by $\bar{x}$ and $\bar{y}$. What can we say about θ?

<u>Example 2</u> For $\bar{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$ find $\cos\theta$. What can we say about θ? (θ is the angle formed by the vectors $\bar{x}$ and $\bar{y}$.)