Linear Algebra – Matrix Transformations & Linear Transformations

Almost all the transformations we will consider are, or can be, associated with matrix multiplication. Hence, we will refer to such transformations as ***matrix transformations***.
For a transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$, typically $T(\mathbf{x})$ will be computed as A**x** where A is an *m x n* matrix. (We frequently denote such a transformation by $\mathbf{x} \mapsto A\mathbf{x}$.)

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, and define a transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ by T(**x**) = A**x**, so that

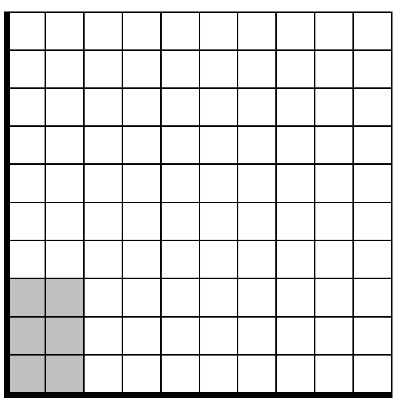
$$T(\mathrm{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}.$$

a. Find $T(\mathbf{u})$.

b. If possible find $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathrm{x}) = \mathbf{b}$.

c. Determine the range of T.

d. If T acts on each point in the rectangle shown below, what do you think the set of images of those points will look like?



u	$T(\mathbf{u})$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	
$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	

A transformation (function or mapping) is ***linear*** provided:
(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of T;
(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all **u** in the domain of T and all scalars c.

Our example is defined by $T(x) = A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$.

e. Is the transformation T defined above a linear transformation?

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and c be a scalar. Calculate $T(\mathbf{u})$, $T(\mathbf{v})$, $T(\mathbf{u} + \mathbf{v})$, $T(\mathbf{u}) + T(\mathbf{v})$, $T(c\mathbf{u})$ and $cT(\mathbf{v})$. Next compare $T(\mathbf{u} + \mathbf{v})$ and $T(\mathbf{u}) + T(\mathbf{v})$; then compare $T(c\mathbf{u})$ and $cT(\mathbf{v})$.

$T(\mathbf{u}) =$; $T(\mathbf{v}) =$; $T(\mathbf{u}) + T(\mathbf{v}) =$

$T(\mathbf{u} + \mathbf{v}) =$

$cT(\mathbf{u}) =$; $T(c\mathbf{u}) =$

f. Is the transformation $F: R^3 \to R^2$ defined by $F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_3^2 \end{bmatrix}$ a linear transformation?