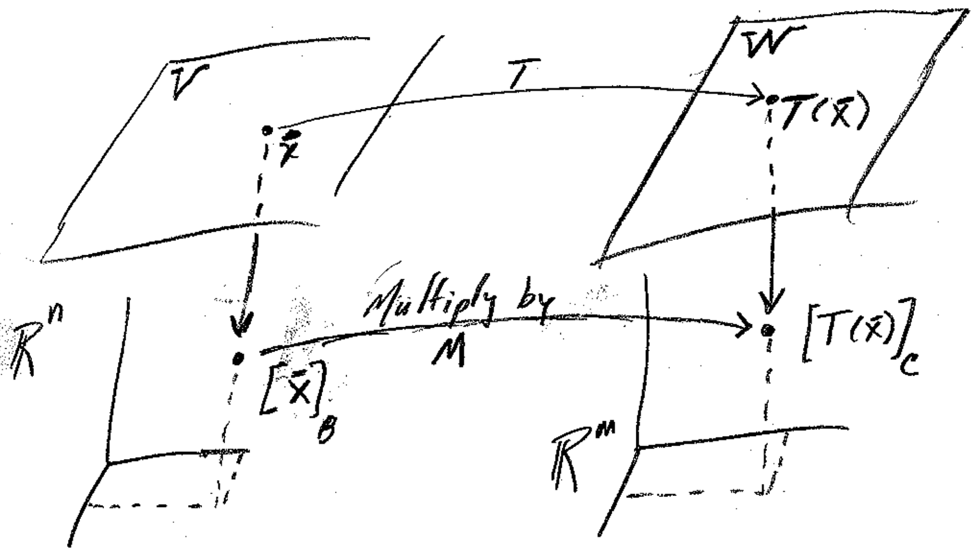
The Matrix of a Linear Transformation

Suppose ✓ V is an n-dimensional vector space with basis $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$

✓ W is an m-dimensional vector space with basis $C = \{\bar{c}_1, \dots, \bar{c}_m\}$.

✓ T is a linear transformation $T: V \to W$



Take $\bar{x} \in V$, $\bar{x} = r_1\bar{b}_1 + r_2\bar{b}_2 + \dots + r_n\bar{b}_n \Rightarrow [\bar{x}]_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$

$$T(\bar{x}) = r_1 T(\bar{b}_1) + r_2 T(\bar{b}_2) + \dots + r_n T(\bar{b}_n)$$

$$[T(\bar{x})]_C = r_1 [T(\bar{b}_1)]_C + r_2 [T(\bar{b}_2)]_C + \dots + r_n [T(\bar{b}_n)]_C$$

$$= \left[[T(\bar{b}_1)]_C \;\vdots\; [T(\bar{b}_2)]_C \;\vdots\; \dots \;\vdots\; [T(\bar{b}_n)]_C \right] \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

$= M[\bar{x}]_B$ where the i^{th} column of M is $[T(\bar{b}_i)]_C$

M is the matrix for T relative to the bases $\mathcal{B}$ and C

Consider $T: V \to V$ where T is a linear transformation and V is a vector space with basis $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ the matrix M in this case is called the matrix for T relative to $\mathcal{B}$, or the $\mathcal{B}$-matrix for T and is denoted by $[T]_{\mathcal{B}}$.

It follows that

$$[T(\bar{x})]_B = [T]_B [\bar{x}]_B$$

Example from 11/30

$T: \mathbb{P}_2 \to \mathbb{P}_2$ is defined by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$.

T is a linear transformation

Calculate images of basis vectors

$$T(1) = 0 \,;\; T(t) = 1 \,;\; T(t^2) = 2t$$

Map to $\mathcal{B}$-coordinate vectors

$$[T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};\; [T(t)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\; [T(t^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Form $\mathcal{B}$-matrix for T

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

For arbitrary $\bar{p}(t) = a_0 + a_1 t + a_2 t^2$

$$[T(\bar{p})]_B = [a_1 + 2a_2 t]_B = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [T]_B [\bar{p}]_B$$

<u>Theorem 8</u> Suppose $A = PDP^{-1}$ where is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\bar{x} \to A\bar{x}$.

<u>Proof</u> Let $P = [\bar{b}_1 \,\vdots\, \bar{b}_2 \,\vdots\, \cdots \,\vdots\, \bar{b}_n]$. By previous work we know $P = P_B$ is the change-of-coordinates matrix such that $P[\bar{x}]_B = \bar{x}$ and $[\bar{x}]_B = P^{-1}\bar{x}$.

Suppose $T(\bar{x}) = A\bar{x}$ for $\bar{x} \in \mathbb{R}^n$. We know that

$$[T]_B = \left[[T(\bar{b}_1)]_B \,\vdots\, \cdots \,\vdots\, [T(\bar{b}_n)]_B \right] \quad \text{, defn } [T]_B$$

$$= \left[[A\bar{b}_1]_B \,\vdots\, \cdots \,\vdots\, [A(\bar{b}_n)]_B \right] \quad , \ T(\bar{x}) = A\bar{x}$$

$$= \left[P^{-1}A\bar{b}_1 \,\vdots\, \cdots \,\vdots\, P^{-1}A\bar{b}_n \right] \quad ,$$

$$= P^{-1}A[\bar{b}_1 \,\vdots\, \cdots \,\vdots\, b_n]$$

$$= P^{-1}AP$$

$$= P^{-1}(PDP^{-1})P$$

$$= D$$