The Matrix of a Linear Transformation

Suppose $T:\mathrm{R}^3 \rightarrow \mathrm{R}^2$ is defined as follows:

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 4x_2 + 6x_3 \\ x_2 + 3x_3 \end{bmatrix} \text{ for all } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathrm{R}^3.$$

1. Calculate each of the following:

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) =$$

2. Find a 2×3 matrix A such that $T(\mathrm{x}) = \mathrm{A}\mathbf{x}$ for each $\mathbf{x} \in \mathrm{R}^3$.

3. Show how to calculate $T\left(\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}\right)$ three different ways.

The special $n \times n$ matrix $\mathrm{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is called an ***identity matrix*** and its columns are all vectors in R^n and are denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ respectively. So, $\mathbf{e}_j$ is the column vector with j^{th} entry 1 and all other entries zero.

Theorem 10. Suppose $T:\mathrm{R}^n \rightarrow \mathrm{R}^m$ is a linear transformation. There exists a unique $m \times n$ matrix A such that $T(\mathrm{x}) = \mathrm{A}\mathbf{x}$ for all $\mathbf{x} \in \mathrm{R}^3$. A is the matrix whose j^{th} colum is the vector $T(\mathbf{e}_j)$. So, $\mathrm{A} = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n)\,]$. A is called the ***standard matrix for the linear transformation*** T.

4. Find the standard matrix for the linear transformation $T:\mathrm{R}^2 \rightarrow \mathrm{R}^2$ defined by

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -2x_1 \end{bmatrix}.$$

5. Find and graph the image of the unit square under transformation of #4 above.

A transformation (mapping) T: $\mathbf{R}^n \to \mathbf{R}^m$ is *onto* $\mathbf{R}^m$ if each $\mathbf{b} \in \mathbf{R}^m$ is the image of at least one $\mathbf{x} \in \mathbf{R}^n$.

6. Is the mapping T: $\mathbf{R}^3 \to \mathbf{R}^2$ defined by
$T(\mathbf{x}) = T([x_1, x_2, x_3]) = [2x_1 + 4x_2 + 6x_3, x_2 + 3x_3]$ for all $\mathbf{x} = [x_1, x_2, x_3] \in \mathbf{R}^3$ onto R^2?

A transformation (mapping) T: $\mathbf{R}^n \to \mathbf{R}^m$ is *one-one* if each $\mathbf{b} \in \mathbf{R}^m$ is the image of at most one $\mathbf{x} \in \mathbf{R}^n$.

7. Is the mapping T: $\mathbf{R}^3 \to \mathbf{R}^2$ defined by
$T(\mathbf{x}) = T([x_1, x_2, x_3]) = [2x_1 + 4x_2 + 6x_3, x_2 + 3x_3]$ for all $\mathbf{x} = [x_1, x_2, x_3] \in \mathbf{R}^3$ one-one?

8. Suppose T: $\mathbf{R}^n \to \mathbf{R}^m$ is defined by $T(\mathbf{x}) = \mathbf{Ax}$ for the $m \times n$ matrix **A**. What can we conclude about T if

   a. the columns of **A** span $\mathbf{R}^m$
   b. the columns of **A** don't span $\mathbf{R}^m$
   c. the columns of **A** are linearly independent
   d. the columns of **A** are dependent
   e. $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution
   f. $\mathbf{Ax} = \mathbf{0}$ has many solutions
   g. The matrix **A** has a pivot in each row
   h. The matrix **A** has a pivot in each column
   i. $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbf{R}^m$
   j. $\mathbf{Ax} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbf{R}^m$
   k. $\mathbf{Ax} = \mathbf{b}$ is consistent and has a unique solution for all $\mathbf{b} \in \mathbf{R}^m$
   l. The linear system with augmented matrix [**A** : **b**] has free variables
   m. The linear system with augmented matrix [**A** : **b**] has no free variables
   n. The augmented matrix [**A** : **b**] has a pivot in each row
   o. The augmented matrix [**A** : **b**] has a pivot in each column