Vectors & Matrices

✓ Vectors in $\mathbb{R}^n$ $\quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

✓ vector addition $\quad \bar{v} + \bar{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}$

✓ scalar multiplication $\quad \alpha \bar{v} = \begin{bmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{bmatrix}$ for $\alpha \in \mathbb{R}$

✓ $m \times n$ matrix $\quad \mathbb{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [\bar{a}_1, \cdots, \bar{a}_n] = [a_{ij}]_{m \times n}$

✓ product of $\mathbb{A}$ and $\bar{v}$ $\quad \mathbb{A}\bar{v} = v_1\bar{a}_1 + v_2\bar{a}_2 + \dots + v_n\bar{a}_n$

$$= [c_{i1}]_{n \times 1} \quad \text{where } c_{i1} = \sum_{k=1}^{m} a_{ik} v_k \quad i = 1, \dots, n$$

✓ matrix addition of $m \times n$ matrices $\mathbb{A}$ and $\mathbb{B}$

$$\mathbb{A} + \mathbb{B} = [\bar{a}_1 + \bar{b}_1, \bar{a}_2 + \bar{b}_2, \cdots, \bar{a}_n + \bar{b}_n] = [a_{ij} + b_{ij}]_{m \times n}$$

for $i = 1, \dots, m$
$j = 1, \dots, n$

✓ scalar multiplication $\alpha \mathbb{A}$

$$\alpha \mathbb{A} = [\alpha \bar{a}_1, \alpha \bar{a}_2, \dots, \alpha \bar{a}_n] = [\alpha a_{ij}] \quad \text{for } \alpha \in \mathbb{R}$$

$i = 1, \dots, m$
$j = 1, \dots, n$

✓ product of $m \times p$ matrix $\mathbb{A}$ and $p \times n$ matrix $\mathbb{B}$

$\mathbb{A}\mathbb{B} = \mathbb{C}$ where $\mathbb{C}$ is $m \times n$ and

$$\mathbb{C} = [\mathbb{A}\bar{b}_1, \mathbb{A}\bar{b}_2, \dots, \mathbb{A}\bar{b}_n] = [c_{ij}]_{m \times n} \quad \text{where } c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$i = 1, \dots, m$
$j = 1, \dots, n$

✓ Linear Systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases}$$

✓ Augmented Matrix for a Linear System

$$\begin{bmatrix} A & \bar{b} \end{bmatrix} = [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n, \bar{b}]$$

✓ Using Row Reduction to Solve a Linear System [p.24]

✓ Existence & Uniqueness Theorem [p.24]

✓ linear combination of vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$

$$\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \cdots + \alpha_k\bar{v}_k \quad \text{where } \alpha_i \in \mathbb{R},\ i = 1, \ldots, k$$

✓ parametric vector form for the solution set of a consistent linear system.

✓ linearly independent (dependent) set of vectors

✓ linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

$$T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$$
$$T(\alpha\bar{u}) = \alpha T(\bar{u})$$

✓ matrix representation for a linear transformation

✓ composition of linear transformations

✓ onto and one-one transformations

✓ properties of vector & matrix operations and special vectors and matrices.

Important Properties

Algebraic Properties of $\mathbb{R}^n$

For all $\bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$:

(i) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$

(ii) $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$

(iii) $\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}$ where $\bar{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

(iv) $\bar{u} + (-\bar{u}) = (-\bar{u}) + (\bar{u}) = \bar{0}$ where $-\bar{u} = -1\bar{u}$

(v) $\alpha(\bar{u} + \bar{v}) = \alpha\bar{u} + \alpha\bar{v}$

(vi) $(\alpha + \beta)\bar{u} = \alpha\bar{u} + \beta\bar{u}$

(vii) $\alpha(\beta\bar{u}) = (\alpha\beta)\bar{u}$

(viii) $1\bar{u} = \bar{u}$

Properties of Matrix-Vector Product

For A $m \times n$ and $\bar{u}, \bar{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$:

(i) $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v}$

(ii) $A(\alpha\bar{u}) = \alpha A\bar{u}$

Properties of Matrix Algebra

For A, B, C of the same size and $\alpha, \beta \in \mathbb{R}$

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$, where O is a matrix of 0's.

(iv) $\alpha(A + B) = \alpha A + \alpha B$

(v) $(\alpha + \beta)A = \alpha A + \beta A$

(vi) $\alpha(\beta A) = (\alpha\beta)A$

Properties of Matrix Multiplication

For matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}$ where the sums and products are defined and $\alpha \in \mathbb{R}$,

(i) $\mathbb{A}(\mathbb{B}\mathbb{C}) = (\mathbb{A}\mathbb{B})\mathbb{C}$

(ii) $\mathbb{A}(\mathbb{B}+\mathbb{C}) = \mathbb{A}\mathbb{B} + \mathbb{A}\mathbb{C}$

(iii) $(\mathbb{B}+\mathbb{C})\mathbb{A} = \mathbb{B}\mathbb{A} + \mathbb{C}\mathbb{A}$

(iv) $\alpha(\mathbb{A}\mathbb{B}) = (\alpha\mathbb{A})\mathbb{B} = \mathbb{A}(\alpha\mathbb{B})$

(v) $\mathbb{I}_m \mathbb{A} = \mathbb{A} = \mathbb{A}\mathbb{I}_n$ where $\mathbb{I}_k$ is identity matrix of appropriate size.

(vi) $(\mathbb{A}^T)^T = \mathbb{A}$

(vii) $(\mathbb{A}+\mathbb{B})^T = \mathbb{A}^T + \mathbb{B}^T$

(viii) $(\alpha\mathbb{A})^T = \alpha\mathbb{A}^T$

(ix) $(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T$