The Concept of a Vector Space
An Abstract Mathematical Structure

A ***vector space*** is a nonempty set V of objects called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms listed below. The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.

1. The sum of **u** and **v**, denoted by **u** + **v**, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector **0** in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each **u** in V, there is a vector –**u** in v such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of **u** by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Example1 : The space R^3 is an example of a vector space. In fact, the space R^n, for any $n \geq 1$, is also an example of a vector space.

Example 2: Let $P_3 = \{a_0 + a_1t + a_2t^2 + a_3t^3 : a_0, a_1, a_2, a_3 \in R\}$. P_3 is the set of all polynomials of degree at most 3. Of course these polynomials have the form

$$\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Where the coefficients a_0, a_1, a_2, a_3 are real numbers and the variable t can assume real values.

Example 3. Let V be the set of real valued functions defined on some interval D on the real line. Functions are added in the usual way: **f** + **g** is the function whose value at $t \in D$ is **f**(t) + **g**(t). Also, for a scalar c and **f**, c**f** is the function whose value at t is c**f**(t). If D = R and **f**(t) = cos(t) + 3 and **g**(t) = $4t^2 + 5t + 6$, then

(**f** + **g**)(t) =   and (5**g**)(t) =

Two functions are equal iff their values are equal for each $t \in D$.

Exercise 1: Describe at least two subspaces of each of the vector spaces in the examples above.

Exercise 2: Suppose $\mathbf{v_1}$ and $\mathbf{v_2}$ are the following vectors in P_3 as defined in Example 2.

$$\mathbf{v_1}(t) = 2 \qquad \text{and} \qquad \mathbf{v_2}(t) = t^2$$

Let H = Span { $\mathbf{v_1}$, $\mathbf{v_2}$ }. Show that H is a subspace of P_3.

Recall the definition of a linear transformation.

> A transformation (function or mapping) is ***linear*** provided:
> (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T;
> (ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in the domain of T and all scalars c.

Exercise 3. Consider the mapping D: $P_3 \rightarrow P_3$ defined by $D(\mathbf{f}) = \frac{df}{dt}$. Is D a linear transformation?

The **kernel** (or **null space**) of a linear transformation T is the set of all $\mathbf{v}$ in the domain of T such that $T(\mathbf{v}) = \mathbf{0}$ (the zero vector in the codomain of T).

Exercise 4. Identify the kernel of the transformation D of Exercise 3.