Linear Algebra (Session # 33)

Theorem Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\}$ be a basis for a vector space V. The coordinate mapping $\bar{x} \mapsto [\bar{x}]_B$ is a 1-1 linear transformation from V onto $\mathbb{R}^n$. (The mapping is an isomorphism)

Proof Take $\bar{u}, \bar{v} \in V$ then there exists scalars c_i, d_i $(i = 1, 2, \ldots, n)$ such that

$$\bar{u} = c_1\bar{b}_1 + \cdots + c_n\bar{b}_n \quad \text{and}$$
$$\bar{v} = d_1\bar{b}_1 + \cdots + d_n\bar{b}_n$$

$$[\bar{u}+\bar{v}]_B = [(c_1\bar{b}_1 + \cdots + c_n\bar{b}_n) + (d_1\bar{b}_1 + \cdots + d_n\bar{b}_n)]_B$$
$$= [(c_1+d_1)\bar{b}_1 + \cdots + (c_n+d_n)\bar{b}_n]_B$$
$$= \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\bar{u}]_B + [\bar{v}]_B$$

So, the mapping preserves addition.

Now, if k is any scalar,

$$[k\bar{u}]_B = [k(c_1\bar{b}_1 + \cdots + c_n\bar{b}_n)]_B$$
$$= [(kc_1)\bar{b}_1 + \cdots + (kc_n)\bar{b}_n]_B$$
$$= \begin{bmatrix} kc_1 \\ \vdots \\ kc_n \end{bmatrix} = k\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = k\,[\bar{u}]_B$$

So, the mapping preserves scalar multiplication.

We have shown the coordinate mapping is a linear transformation.

We will now show the mapping is 1-1.

Suppose $[\bar{u}]_B = [\bar{v}]_B$ for some $\bar{u}, \bar{v} \in V$.

Let $[\bar{u}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and $[\bar{v}]_B = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$

It follows that $c_i = d_i$ for $i = 1, 2, \dots, n$

Hence $\bar{u} = c_1 \bar{b}_1 + \dots + c_n \bar{b}_n$

$= d_1 \bar{b}_1 + \dots + d_n \bar{b}_n$

$= \bar{\bar{v}}$

So, $[\bar{u}]_B = [\bar{v}]_B \Rightarrow \bar{u} = \bar{v}$.

Hence, $\bar{x} \mapsto [\bar{x}]_B$ is 1-1.

We will now show the mapping is onto.

Take $\bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ and consider $\bar{u} \in V$

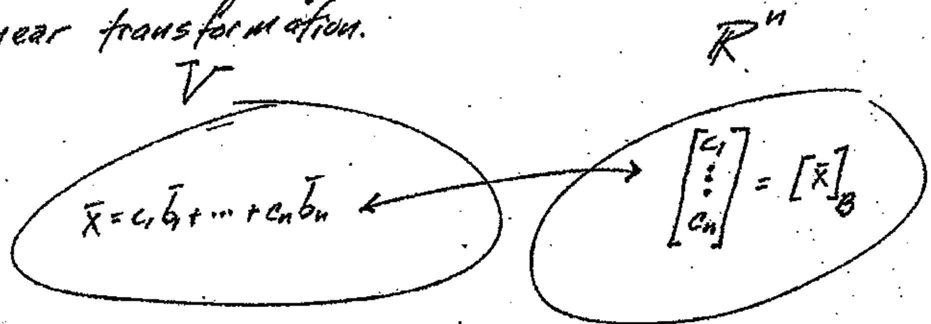
such that $\bar{u} = y_1 \bar{b}_1 + \dots + y_n \bar{b}_n$.

Clearly, $[\bar{u}]_B = \bar{y}$ and hence $\bar{x} \mapsto [x]_B$ is onto.

Consequently, $\bar{x} \mapsto [x_B]$ is a 1-1, onto linear transformation from V to $\mathbb{R}^n$.

Linear Algebra -

Theorem Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n\}$ be a basis for a vector space V. The coordinate mapping $\bar{x} \mapsto [\bar{x}]_{\mathcal{B}}$ is a one-to-one linear transformation.



Proof (In class notes)

Remark A coordinate mapping is an important example of an isomorphism from V onto $\mathbb{R}^n$. A 1-1 linear transformation from a vector space V onto a vector space W is called an isomorphism, and if such a transformation exists V and W are called isomorphic.

Example Suppose $\bar{b}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\bar{b}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ is a basis for $\mathbb{R}^2$. For $\bar{x} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \in \mathbb{R}^2$ specify $[\bar{x}]_{\mathcal{B}}$.

— If $B = [\bar{b}_1 \vdots \bar{b}_2]$ we seek the coordinate vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\bar{x}]_{\mathcal{B}}$ such that $\boxed{B[\bar{x}]_{\mathcal{B}} = \bar{x}.}$ That is, we solve

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$. So in this case $\begin{bmatrix} 8 \\ 9 \end{bmatrix}_{\mathcal{B}} =$

The matrix $\mathbb{B} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is the change-of-coordinates matrix from $\mathbb{B}$ to the standard basis in $\mathbb{R}^n$. We usually denote this matrix by $\mathbb{P}_B$. Hence,

$$\boxed{\mathbb{P}_B [\bar{x}]_B = \bar{x}}$$

Let's calculate $\mathbb{P}_B^{-1}$. $\mathbb{P}_B^{-1} =$

Of course we have

$$\boxed{\mathbb{P}_B^{-1} \bar{x} = [\bar{x}]_B}$$

Use either $\mathbb{P}_B$ or $\mathbb{P}_B^{-1}$ as appropriate to calculate the following missing vectors.

a) $\begin{bmatrix} \\ \end{bmatrix}_B = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ b) $\begin{bmatrix} -8 \\ 15 \end{bmatrix}_B = \begin{bmatrix} \\ \end{bmatrix}$ c) $\begin{bmatrix} 12 \\ \end{bmatrix}_B = \begin{bmatrix} 8 \\ \end{bmatrix}$

Example Suppose $\bar{c}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\bar{c}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$. Consider the coordinate mapping $\bar{x} \mapsto [\bar{x}]_C$.

a) If $[\bar{x}]_C = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, then $\bar{x} =$

b) If $\bar{x} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, then $[\bar{x}]_C =$

c) If $[\bar{x}]_C = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, then $[\bar{x}]_B =$ and $\bar{x} =$

d) If $[\bar{x}]_B = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$, then $[\bar{x}]_C =$ and $\bar{x} =$