Let us work to make correct use of the vocabulary of linear algebra.
Let us work to avoid improper use of the vocabulary of linear algebra.

Systems of linear equations, vector equations, and matrix equations have solutions, basic variables, and maybe free variables. A matrix by itself is neither a system of linear equations, vector equation, nor a matrix equation. Hence, we do not speak of a matrix as having a solution, basic variables, or free variables. Also, we do not speak of a matrix as being linearly independent or dependent. We can speak of the columns of a matrix being linearly independent or dependent.

Example of a linear system

$$\begin{aligned} 1x_1 + 2x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 + x_3 &= 19 \end{aligned}$$

Corresponding vector equation

$$x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}2\\5\end{bmatrix} + x_3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}8\\19\end{bmatrix}$$

Corresponding matrix equation

$$\begin{bmatrix}1 & 2 & 1\\2 & 5 & 1\end{bmatrix}\overline{x} = \begin{bmatrix}8\\19\end{bmatrix}$$

Matrix of coefficients reduced to rref

$$\begin{bmatrix}1 & 2 & 1\\2 & 5 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & -1\end{bmatrix}$$ pivot columns: 1 & 2

Augmented matrix reduced to rref

$$\begin{bmatrix}1 & 2 & 1 & 8\\2 & 5 & 1 & 19\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 2\\0 & 1 & -1 & 3\end{bmatrix}$$ pivot columns: 1 & 2

With respect to the linear system: basic variables: x_1, x_2 ; free variable: x_3

The linear system is consistent because the rightmost column of its augmented matrix is not a pivot column. Since there are free variables, the system has infinitely many solutions. Theorem 3 on page 42 of our text establishes the relationship between a linear system, a vector equation, and a matrix equation.

By Theorem 13 on page 172 of our text, the pivot columns of $\begin{bmatrix}1 & 2 & 1\\2 & 5 & 1\end{bmatrix}$ form a basis for

$Col \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$. Hence, $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ is a basis for $Col \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$. Moreover, as a consequence of the Theorem 15 on page 179, B is also a basis for R^2.

Consider the linear transformation T: $\mathrm{R}^3 \xrightarrow{\text{into}} \mathrm{R}^2$ defined by $T(\bar{x}) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix} \bar{x}$.

What is dim domain T? Is T onto R^2? Is T 1-1? What is the range of T?

Describe $Col \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$. What is rank $Col \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$? What is dim $Col \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$?

Describe $Nul \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$.

What is dim $Nul \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$?

Suppose A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}$. We have seen above that the set $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ is *a* basis for *Col* A. Since there are many other sets that are also bases for *Col* A, we do not refer to *the* basis for *Col* A. Also, although we may say "$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ *form* a basis," we don't say "$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ *is* a basis."

Suppose B = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. We can say "the columns of B are linearly independent." However, we don't say B is linearly independent.