$\boxed{\frac{5.4}{\#6}}$ $T: \mathbb{P}_2 \to \mathbb{P}_4$ is defined by $T(\bar{p}(t)) = \bar{p}(t) + t^2 \bar{p}(t)$.

a) If $\bar{p}(t) = 2 - t + t^2$, then

$$T(\bar{p}(t)) = 2 - t + t^2 + 2t^2 - t^3 + t^4$$

$$\Rightarrow \boxed{T(\bar{p}(t)) = 2 - t + 3t^2 - t^3 + t^4}$$

b) Take $c \in \mathbb{R}$, $\bar{p}(t) = \mathbb{P}_2$

$$T(c\bar{p}(t)) = c\bar{p}(t) + ct^2 \bar{p}(t)$$
$$= c\left(p(t) + t^2 p(t)\right)$$
$$= c\, T(\bar{p}(t))$$

So scalar multiplication is preserved

Take $\bar{p}(t), \bar{q}(t) \in \mathbb{P}_2$

$$T(\bar{p}(t) + \bar{q}(t)) = (\bar{p}(t) + \bar{q}(t)) + t^2(\bar{p}(t) + \bar{q}(t))$$
$$= (\bar{p}(t) + t^2 \bar{p}(t)) + (\bar{q}(t) + t^2 \bar{q}(t))$$
$$= T(\bar{p}(t)) + T(\bar{q}(t))$$

So vector addition is preserved

$\Rightarrow T$ is a linear transformation

c) $T(1) = 1 + t^2$

$T(t) = t + t^3$

$T(t^2) = t^2 + t^4$

So the coordinate vectors for the images of $1, t, t^2$ are $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ respectively. So matrix we seek for T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5.4 #10b) $T: P_3 \to R^4$ defined by

$T(\bar{p}) = \begin{bmatrix} \bar{p}(-3) \\ \bar{p}(-1) \\ \bar{p}(1) \\ \bar{p}(3) \end{bmatrix}$ the matrix we seek is.

$$\begin{bmatrix} 1 & -3 & 9 & -27 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix}$$

The columns of the matrix are the images of $1, t, t^2, t^3$ respectively.

5.4 #14 $T: R^2 \to R^2$ is defined by $T(\bar{x}) = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$.

We seek a basis $\mathcal{B}$ for R^2 with the property that $[T]_B$ is diagonal.

We apply Theorem 8 and diagonalize $\begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$.

Find the eigenvalues

$$\det\begin{pmatrix} 5-\lambda & -3 \\ -7 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - 21 = 0$$

$$5 - 6\lambda + \lambda^2 - 21 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda - 8)(\lambda + 2) = 0$$

Our characteristic values are 8 and -2. We find the associated eigenvectors

$\underline{\lambda=8}$ $\begin{bmatrix} -3 & -3 & 0 \\ -7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \bar{v} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda = 8$

$\underline{\lambda=-2}$ $\begin{bmatrix} 7 & -3 & 0 \\ -7 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \bar{v} = t\begin{bmatrix} \frac{3}{7} \\ 1 \end{bmatrix}$

So, $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ is an eigenvector associated with $\lambda = -2$

$\Rightarrow \begin{bmatrix} -1 & 3 \\ 1 & 7 \end{bmatrix}$ diagonalizes $\begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$

So $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}\right\}$ has the property that the $\mathcal{B}$-matrix for T is a diagonal matrix. Moreover

$$[T]_B = \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}.$$

7.1 / #8 The columns of $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ form an orthonormal basis for $\mathbb{R}^2$. So, the matrix is called orthogonal.

Its inverse is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

7.1 / #16 $A = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$; A's eigenvalues are found as follows:

$$\det\begin{pmatrix} -7-\lambda & 24 \\ 24 & 7-\lambda \end{pmatrix} = (\lambda+7)(\lambda-7) - 24^2 = \lambda^2 - 49 - 576 = 0$$

$(\lambda^2 - 25^2) = 0 \Rightarrow \lambda_1 = -25$ and $\lambda_2 = 25$ are the eigenvalues

The corresponding eigenvectors are $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Normalizing them we get an orthonormal basis the columns of $P = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

So, $A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -25 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$